

# Channel Uncertainty in Ultra Wideband Communication Systems

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## Abstract

Wide band systems operating over multipath channels may spread their power over an infinitely wide bandwidth if they use duty cycle. At the limit of infinite bandwidth, direct sequence spread spectrum and pulse position modulation systems with duty cycle achieve the channel capacity, if the increase of the number of channel paths with the bandwidth is not too rapid. The higher spectral efficiency of the spread spectrum modulation lets it achieve the channel capacity in the limit, in environments where pulse position modulation with non-vanishing symbol time cannot be used because of the large number of channel paths. Channel uncertainty limits the achievable data rates of power constrained wide band systems; Duty cycle transmission reduces the channel uncertainty because the receiver has to estimate the channel only when transmission takes place. The optimal choice of the fraction of time used for transmission depends on the spectral efficiency of the signal modulation.

## I. INTRODUCTION

This work discusses the achievable data rates of systems with very wide bandwidths. Considering communication with an average power constraint, the capacity of the multipath channel in the limit of infinite bandwidth is identical to the capacity of the additive white Gaussian noise (AWGN) channel  $C_{\text{AWGN}} = P/N_0 \log e$ , where  $P$  is the average received power and  $N_0$  is the received noise spectral density. *Kennedy* [1] and *Gallager* [2] proved this for fading channels using FSK signals with duty cycle transmission; *Telatar and Tse* [3] extended the proof for multipath channels with any number of paths. The AWGN capacity is achievable on multipath

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channels also by dividing the spectrum into many narrow bands and transmitting bursty signals separately on each band.

When using spreading modulations, *Médard and Gallager* [5] show that direct sequence spread spectrum signals, when transmitted continuously (no duty cycle) over fading channels (that have a very large number of channel paths), approach zero data rate in the limit of infinite bandwidth. A similar result was shown by *Subramanian and Hajek* [6]. *Telatar and Tse* [3] show that over multipath channels, the data rate in the limit of infinite bandwidth is inversely proportional to the number of channel paths.

This work is motivated by a recent surge in interest in ultra wide band systems, where spreading signals are often desired. It shows that under suitable conditions, spreading signals can achieve AWGN capacity on multipath channels in the limit of infinite bandwidth, if they are used with duty cycle. In other words, peakiness in time is sufficient to achieve AWGN capacity, and the transmitted signal does not have to be peaky in frequency as well. We analyze direct sequence spread spectrum (DSSS) and pulse position modulation (PPM) signals, and show that when the scaling of the number of channel paths is not too rapid, these signals achieve the capacity in the limit as the bandwidth grows large.

Our results can be seen as a middle ground between two previous results: 1. FSK with duty cycle achieves AWGN capacity for any number of channel paths and 2. direct sequence spread spectrum signals with continuous transmission (no duty cycle) have zero throughput in the limit, if the number of channel paths increases with the bandwidth.

Our main results are as follows. In the limit of infinite bandwidth, DSSS systems where the receiver knows the path delays achieve AWGN capacity if the number of channel path is sub-linear in the bandwidth, formally if  $\frac{L}{W} \rightarrow 0$  where  $L$  is the number of independently fading channel paths and  $W$  is the bandwidth, and the system uses an appropriate duty cycle. PPM systems too can achieve AWGN capacity in the limit of infinite bandwidth, but this is possible for smaller number of channel paths. A PPM system with a receiver that knows the path delays achieves AWGN capacity if  $\frac{L}{\log W} \rightarrow 0$ . PPM systems with lower bounded symbol time have zero throughput if  $\frac{L}{\log W} \rightarrow \infty$ . In systems where the receiver does not know the path gains or delays, we show that DSSS systems can achieve AWGN capacity if  $\frac{L}{W/\log W} \rightarrow 0$  as the bandwidth increases. Measurements of the number of channel paths vs. bandwidth in Figure 1 show an increase of the number of channel paths that appears to be sub-linear.

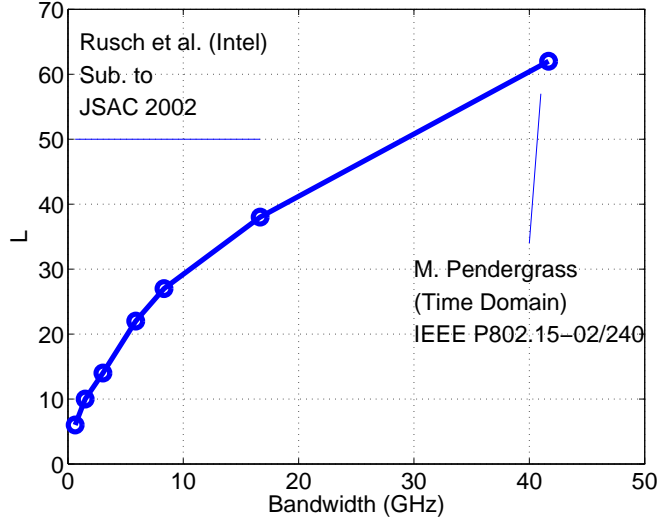


Fig. 1. Number of significant channel paths vs. bandwidth. The paths accounting for 60–90 percent of the energy were counted in two separate measurement campaigns. The Intel data were taken from [7] and the Time Domain data from [8]

The effect of duty cycle can be understood in terms of the channel uncertainty a communication system faces. The data rate is penalized when the receiver has to estimate the channel, so infrequent usage of the channel leads to a small channel uncertainty and a small penalty. The spectral efficiency of the modulation scheme plays an important role in determining the channel uncertainty a system handles. A system with a low spectral efficiency can pack a small number of bits into each transmission period, and in order to maintain a high data rate it must transmit often. Thus, low spectral efficiency forces the communication system to estimate the channel often, and suffer from a large penalty on its data rate.

A useful intuition is given by the ratio

$$\text{SNR}_{\text{est}} = \frac{P T_c}{N_0 \theta L}$$

This ratio compares the channel uncertainty per unit time  $\frac{\theta L}{T_c}$  to the data rate in the limit of infinite bandwidth that is proportional to  $\frac{P}{N_0}$ .  $L$  is the number of independent channel components,  $T_c$  is the coherence time and  $\theta$  is the duty cycle parameter or the fraction of time used for transmission. The ratio  $\text{SNR}_{\text{est}}$  can also be interpreted as the SNR over a coherence period per uncertainty branch of the channel. A communication system can achieve the channel capacity in the limit of infinite bandwidth only if the channel uncertainty per unit time becomes insignificant relative

to the capacity or

$$\text{SNR}_{\text{est}} \rightarrow \infty$$

In systems with bounded average received power, the duty cycle parameter must diminish in order to balance the increase in the number of channel components  $L$ , and let the overall channel uncertainty diminish. Spectrally efficient modulation schemes permit infrequent transmission (or small  $\theta$ ), thus reducing the channel uncertainty per unit time. In contrast, low spectral efficiency forces frequent transmission, and the duty cycle parameter must stay high.

The difference between the wideband capacities of DSSS and PPM schemes comes about precisely because of their different spectral efficiencies. PPM is an orthogonal modulation, so the number of bits it can transmit per unit time increases logarithmically with the bandwidth, where the number of bits a DSSS transmitter can send per unit time increases linearly. Thus, DSSS can tolerate a larger amount of channel uncertainty than PPM. Note that, in contrast, both PPM and DSSS achieve the channel capacity in the limit of infinite bandwidth over the AWGN channel as well as over the multipath fading channel where the channel is known to the receiver.

It is interesting to contrast our results with those of Verdú [4], where the crucial role of peakiness in time is pointed out and a connection is also made between the spectral efficiency of a modulation scheme and its data rate over a large bandwidth. The theory there analyzes the capacity of channels with *fixed* amount of channel uncertainty as the average received power per degree of freedom goes to zero (or, equivalently, the bandwidth goes to infinity). By suitable scaling of the duty cycle, the infinite bandwidth AWGN capacity is always approached in the limit by the modulation schemes considered there, and the issue is the *rate of convergence* to that limit. This rate of convergence depends on the spectral efficiency of the modulation scheme used. In contrast, the environment faced by the modulation schemes considered here is a harsher one, in that the channel uncertainty *increases* with the bandwidth. Now the issue is *whether* a modulation scheme approaches the AWGN capacity at all. The framework in [4] is suitable for systems that break the wideband channel into many narrow-band channels in parallel; one example is the OFDM modulation. In this context, one can focus on a single narrowband channel, in which the channel uncertainty is fixed.

The outline of the paper is as follows. After presenting the channel model and the signals in Section II, a summary of the results is presented in Section III and discussion is brought

in Section IV. Section V presents a bound on the channel uncertainty penalty related to path delays. Sections VI and VII then present bounds on the data rates of DSSS and PPM system where the receiver knows the channel path delays.

## II. CHANNEL MODEL AND SIGNAL MODULATIONS

The natural model for an ultra wide band channel is real, because there is no obvious ‘carrier frequency’ that defines the phase of complex quantities. The channel is composed of  $\tilde{L}$  paths:

$$Y(t) = \sum_{l=1}^{\tilde{L}} A_l(t)X(t - d_l(t)) + Z(t)$$

The received signal  $Y(t)$  is matched filtered and sampled at a rate  $1/W$ , where  $W$  is the bandwidth of the transmitted signal. We get a discrete representation of the received signal. We assume a block fading model: the channel remains constant over coherence periods that last  $T_c$ , and changes independently between coherence periods. The paths delays are in the range  $[0, T_d)$ , where the delay spread  $T_d$  is assumed much smaller than the coherence period, so signal spillover from one coherence period to the next is negligible. Considering a double sided noise density  $\frac{N_0}{2}$  the discretized and normalized signal is given by

$$Y_i = \sqrt{\frac{\mathcal{E}}{K_c}} \sum_{m=1}^{\tilde{L}} A_m X_{i-\tau_m} + Z_i \quad i = 0, \dots, \lfloor T_c W \rfloor - 1 \quad (1)$$

with  $\mathcal{E} = \frac{2PT_c}{N_0\theta} = 2LSNR_{\text{est}}$  and  $K_c = \lfloor T_c W \rfloor$ . The noise  $\{Z_i\}$  is real and Gaussian, and the normalization requires that the path gains  $\{A_m\}$  and the transmitted signal  $\{X_i\}$  are scaled so that  $E(\sum A_m X_{i-m})^2 = 1$ . If  $\{A_m\}$  are independent and zero mean, the scaling requirement is  $\sum E[A_m^2] = 1$  and  $E[X_i^2] = 1$ . This normalization ensures that  $E[Z_i^2] = 1$ .  $P$  is an average received power and  $W$  is the bandwidth.

In order to avoid complications at the edge of the coherence interval, we approximate the channel using a cyclic difference over  $K_c$  instead of a simple difference:  $(n) \equiv n \bmod K_c$ . The difference is negligible as the delay spread is much smaller than the coherence time.

$$Y_i = \sqrt{\frac{\mathcal{E}}{K_c}} \sum_{m=1}^{\tilde{L}} A_m X_{(i-\tau_m)} + Z_i \quad i = 0, \dots, \lfloor T_c W \rfloor - 1 \quad (2)$$

Note that when  $X$  is a PPM signal from Section II-B (that included a  $T_d$  guard time between symbols) the circular formulation (2) is identical to the original model (1).

The path delays  $\{\tau_m\}$  are bunched into resolvable paths, separated by the system time resolution  $\frac{1}{W}$ . The resolvable paths are defined by summing over the paths with similar delays:

$$G_l = \sum_{m: \frac{l}{W} \leq \tau_m < \frac{l+1}{W}} A_m \quad 0 \leq l < \lfloor T_d W \rfloor$$

The number of resolvable paths is  $L$  and their delays  $\{D_l\}_{l=1}^L$  are integers between 0 and  $\lfloor WT_d \rfloor - 1$ . The  $L$  delays are uniformly distributed over the  $\binom{\lfloor WT_d \rfloor}{L}$  possibilities of combinations of  $L$  values out of  $\binom{\lfloor WT_d \rfloor}{L}$  positions.

$$Y_i = \sqrt{\frac{\mathcal{E}}{K_c}} \sum_{l=1}^L G_l X_{i-D_l} + Z_i \quad i = 0, \dots, \lfloor T_c W \rfloor - 1$$

The channel gains are real, we assume that they are IID and independent of the delays. Our normalization requires that the variance of the gains equals  $\frac{1}{L}$ . The delays are assumed IID uniform over  $0, \dots, \lfloor WT_d \rfloor - 1$ .

The systems we consider do not use channel information at the transmitter.

### A. Direct Sequence Spread Spectrum Signals

Each transmitted symbol contains a random series of IID  $K_c$  Gaussian values  $\{x_i\}_{i=0}^{K_c-1}$  with zero mean, and an energy constraint is satisfied:

$$E[X_j^2] = \theta E \left[ \frac{1}{K_c} \sum_{i=0}^{K_c-1} x_i^2 \right] = 1$$

where  $0 < \theta \leq 1$  is the duty cycle parameter or the fraction of time used for transmission. The lower case symbol  $x$  is used for the transmitted signal during active transmission periods, while the upper case symbol  $X$  represents all of the transmitted signal, that includes silent periods due to duty cycle.

We define the autocorrelation of the signal

$$C(m, n) \equiv \frac{\theta}{K_c} \sum_{i=0}^{K_c-1} x_{i-m} x_{i-n} \quad \forall m, n$$

Edge conditions are settled by assuming that the each symbol follows a similar and independent one. For IID chips we have

$$\sum_{m=1}^L \sum_{n=1}^L E_x |C(m, n) - \delta_{mn}| \leq \frac{2L}{\sqrt{\pi K_c}} + \frac{L^2 - L}{K_c} \quad (3)$$

see the proof in the appendix.

The upper bound on DSSS capacity (Section VI-B) is also valid for another type of signals, where  $x$  is composed of pseudo-random sequences of  $K_c$  values. The empirical autocorrelation of the input is bounded and the signal has a delta-like autocorrelation:

$$C(m, n) \equiv \frac{\theta}{K_c} \sum_{i=0}^{K_c-1} x_{i-m} x_{i-n} \quad \forall m, n$$

$$|C(m, n) - \delta(n, m)| \leq \frac{d}{K_c} \quad (4)$$

where  $d$  does not depend on the bandwidth.

In either case (IID or pseudo-random signals) the duty cycle, with parameter  $\theta$ , is used over coherence times: of each period  $\frac{T_c}{\theta}$ , one period of  $T_c$  is used for transmission and in the rest of the time the transmitter is silent.

### B. PPM Signals

The signals defined in this section are used to calculate lower bounds on the data rates of PPM systems (Section VII-A). The upper bound on PPM performance holds for a wider family of signals, defined in Section VII-B.

Signaling is done over periods  $T_s$  long, with  $\lfloor T_s W \rfloor$  positions in each symbol. A guard time of  $T_d$  is taken between the symbols, so the symbol period is  $T_s + T_d$ . The symbol time  $T_s$  is typically in the order of the delay spread or smaller. It does not play a significant role in the results. Each symbol is of the form:

$$x_i = \begin{cases} \sqrt{W(T_s + T_d)} & \text{one position of each group of } \lfloor T_s W \rfloor \\ & \text{with } n\lfloor (T_s + T_d)W \rfloor \leq i \leq n\lfloor (T_s + T_d)W \rfloor + \lfloor T_s W \rfloor - 1 \\ 0 & \text{other positions} \end{cases}$$

$$n = 0, 1, \dots, \left\lfloor \frac{T_c}{T_s + T_d} \right\rfloor - 1$$

$$i = 0, 1, \dots, \lfloor T_c W \rfloor - 1$$

The number of symbols transmitted over a single coherence period is  $N = \frac{T_c}{T_s + T_d}$ . We assume  $N$  is a whole number, this assumption does not alter our the result we prove here. The symbol

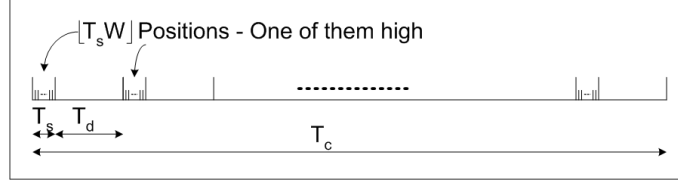


Fig. 2. PPM symbol timing.

timing is illustrated in Figure 2. The duty cycle parameter  $0 < \theta \leq 1$  is used over coherence times: of each period  $\frac{T_c}{\theta}$ , one period of  $T_c$  is used for transmission and in the rest of the time the transmitter is silent.

### III. SUMMARY OF THE RESULTS

The fastest increase of the number of paths that a system can tolerate depends on its spectral efficiency. A direct sequence spread spectrum system with receiver knowledge of the path delays, can tolerate a sub-linear increase of the number of paths with the bandwidth, and achieve AWGN capacity (Section VI-A.1). Conversely, if the number of paths increases linearly with the bandwidth, the data rate is penalized (Sections VI-B).

*Theorem 1: Part 1:* DSSS systems with duty cycle where the receiver knows the path delays (but not necessarily their gains) achieve  $C_{\text{DSSS}} \rightarrow C_{\text{AWGN}}$  as  $W \rightarrow \infty$  if  $\frac{L}{W} \rightarrow 0$ .

*Part 2:* DSSS systems with duty cycle achieve  $C_{\text{DSSS}} \rightarrow C_{\text{AWGN}}$  as  $W \rightarrow \infty$  if  $\frac{L \log W}{W} \rightarrow 0$ . No knowledge of the channel is required.

*Converse 1:* DSSS systems with duty cycle where the path gains are unknown to the receiver and uniformly bounded by  $|G_l| \leq \frac{B}{\sqrt{L}}$  with a constant  $B$ , achieve  $C_{\text{DSSS}} < C_{\text{AWGN}}$  in the limit  $W \rightarrow \infty$  if  $\frac{L}{W} \rightarrow \alpha$  and  $\alpha > 0$ . This bound holds whether the receiver knows the path delays or it does not.

The proof is presented in Section VI.

*Theorem 2:* PPM systems with duty cycle, where the receiver knows the path delays, achieve  $C_{\text{PPM}} \rightarrow C_{\text{AWGN}}$  as  $W \rightarrow \infty$  if  $\frac{L}{\log W} \rightarrow 0$  and the path gains  $\{G_l\}_{l=1}^L$  satisfy (A)  $\max_{1 \leq i \leq L} |G_i| \rightarrow 0$  in probability as  $L \rightarrow \infty$ , and (B)  $E_G = \sum_{i=1}^L G_i^2 \rightarrow 1$  in probability as  $L \rightarrow \infty$ . Note that if the gains are Gaussian IID with zero mean then the above condition holds.

*Converse 2:* PPM systems with a non-vanishing symbol time transmitting over a channel with Gaussian path gains that are unknown to the receiver, achieve  $C_{\text{PPM}} \rightarrow 0$  as  $W \rightarrow \infty$  if  $\frac{L}{\log W} \rightarrow \infty$ . This result holds whether the receiver knows the path delays or it does not. The proof is presented in Section VII.

#### IV. DISCUSSION

This section presents bounds on the data rates of direct sequence spread spectrum and PPM systems for different channels, computed in Sections V, VI, and VII. The channel and system parameters were chosen to represent a realistic low SNR ultra wide band system. For the figures with fixed bandwidth we use: Bandwidth  $W=20$  GHz,  $\frac{P}{N_0}=50$  dB (SNR=-53 dB at  $W=20$  GHz), coherence period  $T_c=0.1$  msec, delay spread  $T_d=200$  nsec, PPM symbol time  $T_s=800$  nsec with guard time of 200 nsec between symbols, and  $B^2d = 1$ .  $B$  is defined in the converse to Theorem 1, it is used to characterize channel gains.  $d$  is defined in (4) for pseudo-random chips; it equals 1 for IID chips. For the figures with a fixed number of paths we use  $L=100$ .

##### A. The Advantage of Duty Cycle

Figure 3 shows the increase of data rate given by the usage of coherence period duty cycle, for DSSS systems. The figure compares the upper bound on DSSS throughput, where duty cycle is not used (bottom graph) to the lower bound on throughput when optimal duty cycle is used. Both bounds decrease as the number of paths  $L$  increases because the channel uncertainty increases as  $L$  increases, and so does the penalty on the data rate.

##### B. The Duty Cycle Parameter

Figure 4 shows the lower bound on the data rate of a direct sequence spread spectrum system for different duty cycle parameter values. The bound is a difference between the data rate of a system with perfect channel knowledge at the receiver and the channel uncertainty penalty (gain penalty and delay penalty). The data rate of a system with perfect channel knowledge equals the channel capacity ( $C_{\text{AWGN}}$ ) in the limit of infinite bandwidth, it is lower when the bandwidth is finite.

The channel uncertainty penalty is small for low values of duty cycle parameter, because the channel is used less often as  $\theta$  decreases. However, the data rate of a system with perfect

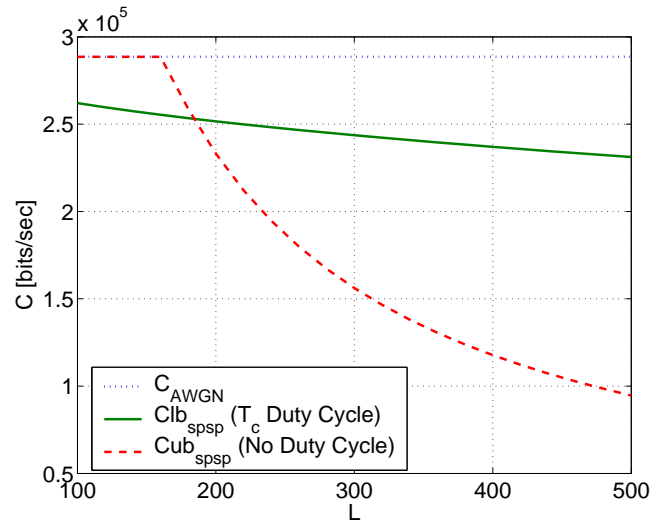


Fig. 3. DSSS throughput bounds, the receiver does not know the channel gains or delays, vs. the number of channel paths. This plot contrasts an upper bound on DSSS data rate, when duty cycle is not used (bottom graph, from (15)) with a lower bound on the data rate when coherence period duty cycle is used (top graph, from (10)).  $C_{AWGN}$  is shown for reference.

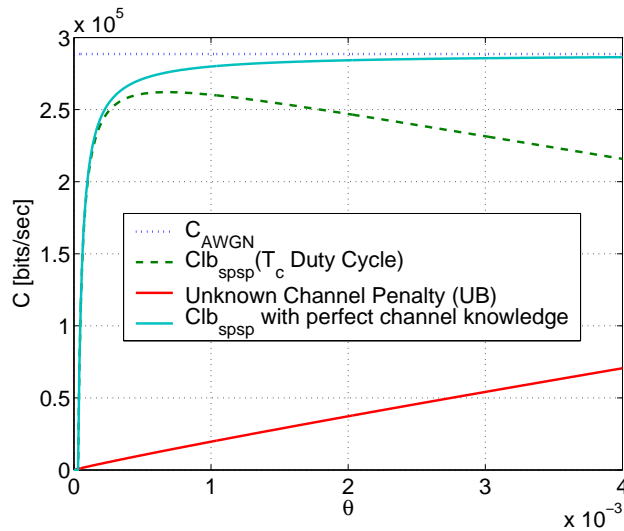


Fig. 4. DSSS throughput lower bound vs. duty cycle parameter, the receiver does not know the channel path gains or delays. The bottom curve shows an upper bound on the channel uncertainty penalty (calculated as the sum of (7) and (5)). The dashed curve shows a lower bound on the system throughput (10), and the top (full) curve shows the throughput of a system with perfect channel knowledge at the receiver (9).  $C_{AWGN}$  is shown at the dotted curve for reference. The system throughput (dashed curve) is the difference between the data rate of a system with perfect channel knowledge at the receiver (top curve) and the channel uncertainty penalty (bottom curve). The throughput is maximized when the spectral efficiency is balanced against the channel uncertainty penalty.

channel knowledge is severely reduced if  $\theta$  is too low. In this case, transmission occurs with high energy per symbol, where the direct sequence spread spectrum modulation is no longer spectrally efficient, so the data rate with perfect channel knowledge is reduced. Figure 4 shows that the duty cycle parameter must be chosen to balance the channel uncertainty penalty (that is large for large  $\theta$ ) and the spectral efficiency of the selected modulation, that increases with  $\theta$ .

### C. Spectral Efficiency

Figure 5 contrasts the achievable data rates of DSSS and PPM systems, when both use duty cycle on coherence periods with optimal duty cycle parameters. Direct sequence spread spectrum achieves higher data rates because it has a higher spectral efficiency, thus it can pack more bits into each transmission period of length  $T_c$ . By packing bits efficiently a DSSS system is able to use a small duty cycle parameter. In contrast, PPM is less efficient in packing bits into its transmission periods, and is thus forced to transmit more often (it has a larger duty cycle parameter, Figure 6). The PPM system is therefore forced to handle a larger number of channel realizations (per unit time), so it suffers a higher penalty for estimating the channel parameters. The number of bits per DSSS symbol depends linearly on the bandwidth, and the number of bits per PPM symbol (with a fixed symbol time) depends logarithmically on the bandwidth, because PPM is an orthogonal modulation.

## V. PATH DELAY UNCERTAINTY PENALTY

The  $L$  path delays are uniformly distributed over the  $\binom{\lfloor WT_d \rfloor}{L}$  possibilities of combinations of  $L$  values out of  $WT_d$  positions spanning the delay spread.

The entropy of the path delays is used to bound the penalty on the mutual information, due to the delay uncertainty. The mutual information in units of [bits/sec]:

$$\begin{aligned}
 I(Y; X) &= I(Y; X, D) - I(Y; D|X) \\
 &\geq I(Y; X|D) - H(D) \\
 &= I(Y; X|D) - \frac{\theta}{T_c} \log_2 \binom{\lfloor WT_d \rfloor}{L} \\
 &\geq I(Y; X|D) - \frac{\theta L}{T_c} \log_2 (WT_d)
 \end{aligned}$$

$\theta$  is the duty cycle parameter, where duty cycle is used over the coherence periods of the channel.

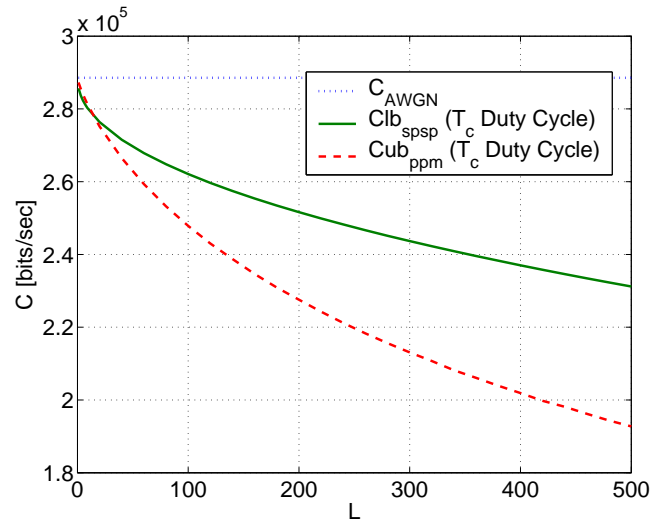


Fig. 5. DSSS and PPM throughput bounds. The DSSS lower bound (10) is calculated without channel knowledge at the receiver. The PPM upper bound (23) is calculated with a receiver that knows the channel delays but not the gains, coherence period duty cycle is used.  $C_{AWGN}$  is shown for reference.

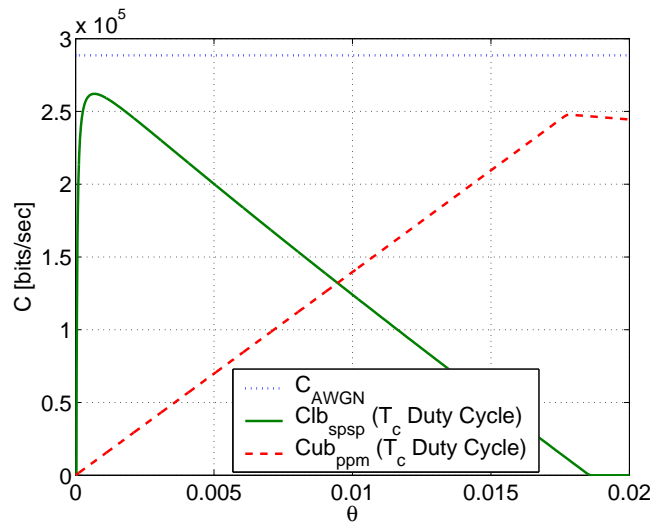


Fig. 6. Throughput bounds vs. duty cycle parameter, the receiver knows the channel path delays but not the gains. DSSS throughput lower bound (10) is maximized at a lower duty cycle parameter than the PPM upper bound (23).

The delay uncertainty penalty is upper bounded:

$$\begin{aligned} \text{Delay Uncertainty Penalty} &= I(Y; D|X) \leq H(D) \\ &\leq \frac{\theta L}{T_c} \log_2(WT_d) \end{aligned} \quad (5)$$

## VI. SPREAD SPECTRUM BOUNDS

### A. When is the Channel Capacity Achieved?

We start with a result in the case of known path delays (Theorem 1 Part 1) that shows that the channel capacity is achieved if the number of paths is sub-linear with the bandwidth. A second result is then given for the case of unknown channel, (Theorem 1 Part 2), that shows that the channel capacity is still achievable, but at simpler environment, with a smaller number of channel paths.

#### 1) Path Delays Known to Receiver:

*Theorem 1: Part 1:* Direct sequence spread spectrum systems with duty cycle, where the receiver knows the path delays, achieve  $C_{\text{DSSS}} \rightarrow C_{\text{AWGN}}$  as  $W \rightarrow \infty$  if  $\frac{L}{W} \rightarrow 0$ .

*Proof:* The proof is based on a lower bound on the mutual information.

*Proposition 3:* DSSS systems where the receiver knows the channel path delays achieve

$$\begin{aligned} I(X; Y|D) \text{ [b/s]} &\geq C_{\text{AWGN}} \\ &- \min_{0 < \theta \leq 1} \left\{ \frac{\theta L}{T_c} \log_2 \left( 1 + \frac{P T_c}{N_0 \theta L} \right) + \frac{3P^2}{N_0^2 \theta W} \log_2 e \right\} \end{aligned} \quad (6)$$

*Discussion of Proposition 3:* The channel uncertainty penalty (due to path gains) has two parts, the first

$$\frac{\theta L}{T_c} \log_2 \left( 1 + \frac{P T_c}{N_0 \theta L} \right) \quad (7)$$

is the penalty due to the unknown gains, it increases as the number of paths  $L$  or the duty cycle parameter  $\theta$  increase.

The second part of the penalty

$$\frac{P^2}{N_0^2 \theta W} \log_2 e$$

is due to the limitation on spectral efficiency of the spread spectrum modulation. It penalizes the system for using a too small duty cycle parameter, where the system concentrates too much energy on each transmitted symbol. Mathematically, this term is the quadratic term in the series approximating the mutual information, that is logarithmic. The first (linear) term in this series equals  $C_{\text{AWGN}}$  in the limit. The balance between the two penalties is shown in Figure 4.

Looking at the limit of infinite bandwidth, the data rate equals the AWGN capacity if  $\theta W \rightarrow \infty$  and  $\theta L \rightarrow 0$ . If  $L$  is sub-linear in  $W$ , these two requirements can be met simultaneously (by choosing the appropriate duty cycle parameter  $\theta$ ). Thus, the proof of Theorem 1 Part 1 follows from Proposition 3.

*Proof of Proposition 3:* The proof of (6) follows Theorem 3 of [3], with a real channel instead of the complex channel used there.

$$I(X; Y|D) \geq I(Y; X|G, D) - I(Y; G|X, D) \quad (8)$$

The first part of (8):

$$I(Y; X|G, D) = \frac{1}{2} E_{G,D} \log \det \left( I + \frac{\mathcal{E}}{WT_c} AA^* \right)$$

where  $A$  is a  $K_c \times K_c$  matrix,  $A_{im} = G_l$  if  $m = (i - D_l)$  and zero otherwise, and  $\mathcal{E} = \frac{2PT_c}{N_0\theta}$ . The eigenvalues of  $AA^*$  are  $\left| F\left(\frac{k}{K_c}\right) \right|^2$ ,  $k = 0, 1, \dots, K_c - 1$ , and

$$F(f) = \sum_{l=1}^L G_l \exp(2\pi j D_l f)$$

For large  $L$  ( $\gg 1$ )  $F(f)$  is complex Gaussian with independent real and imaginary parts that may have different variances (for small  $f$ ).

$$F(f) \sim N(0, 1)$$

so  $E_G |F(f)|^2 = 1$  and  $E_G |F(f)|^4 \leq 3$ .

$$\begin{aligned}
I(Y; X|G, D) &= \frac{1}{2} E_{G,D} \left[ \sum_{k=0}^{WT_c-1} \log \left( 1 + \frac{\mathcal{E}}{WT_c} \left| F \left( \frac{k}{WT_c} \right) \right|^2 \right) \right] \\
&\geq \frac{\log e}{2} E_{G,D} \sum_{k=0}^{WT_c-1} \frac{\mathcal{E}}{WT_c} \left| F \left( \frac{k}{WT_c} \right) \right|^2 \\
&\quad - \frac{1}{2} \frac{\mathcal{E}^2}{W^2 T_c^2} \left| F \left( \frac{k}{WT_c} \right) \right|^4 \\
&\geq \frac{\mathcal{E}}{2} \log e - \frac{3\mathcal{E}^2}{4WT_c} \log e \\
&= \frac{PT_c}{N_0\theta} \log e - \frac{3P^2 T_c}{N_0^2 \theta^2 W} \log e
\end{aligned}$$

In [bits/sec]:

$$I(Y; X|G, D) \geq \frac{P}{N_0} \log_2 e - \frac{3P^2}{N_0^2 \theta W} \log_2 e \quad (9)$$

where we used  $\log(1+x) \geq \left(x - \frac{x^2}{2}\right) \log e$ .

The second part of (8):

$$I(Y; G|X, D) \leq \frac{1}{2} E_{X,D} \log \det \left( I + \frac{\mathcal{E}}{WT_c} B \Lambda B^* \right)$$

where  $B_{im} = X_{(i-m)}$  and  $\Lambda = \frac{1}{L} I$ . The upper bound is tight for a Gaussian channel gains.

Following [3] we get an upper bound

$$I(Y; G|X, D) \leq \frac{L}{2} \log \left( 1 + \frac{\mathcal{E}}{L} \right)$$

Rewriting (8):

$$\begin{aligned}
I(X; Y|D) &\geq I(Y; X|G, D) - I(Y; G|X, D) \\
&\geq \frac{\mathcal{E}}{2} \log e - \frac{3\mathcal{E}^2}{WT_c} \log e - \frac{L}{2} \log \left( 1 + \frac{\mathcal{E}}{L} \right)
\end{aligned}$$

in [bits/sec]:

$$\begin{aligned}
I(X;Y|D) \text{ [b/s]} &= \frac{\theta I(X;Y|D)}{T_c} \\
&\geq \frac{\theta \mathcal{E}}{2T_c} \log_2 e - \frac{\theta L}{2T_c} \log_2 \left( 1 + \frac{\mathcal{E}}{L} \right) \\
&\quad - \frac{3\theta \mathcal{E}^2}{4K_c T_c} \log_2 e \\
&= \frac{P}{N_0} \log_2 e - \frac{\theta L}{2T_c} \log_2 \left( 1 + \frac{P}{N_0} \frac{2T_c}{\theta L} \right) \\
&\quad - \frac{3P^2}{N_0^2 \theta W} \log_2 e
\end{aligned}$$

The bound is valid for any  $\theta$ , and we choose its maximal value:

$$\begin{aligned}
I(X;Y|D) \text{ [b/s]} &\geq \\
&\max_{0 < \theta \leq 1} \left\{ \frac{P}{N_0} \log_2 e \right. \\
&\quad \left. - \frac{\theta L}{T_c} \log_2 \left( 1 + \frac{P}{N_0} \frac{T_c}{\theta L} \right) - \frac{P^2}{N_0^2 \theta W} \log_2 e \right\} \\
&= C_{\text{AWGN}} \\
&\quad - \min_{0 < \theta \leq 1} \left\{ \frac{\theta L}{T_c} \log_2 \left( 1 + \frac{P}{N_0} \frac{T_c}{\theta L} \right) + \frac{P^2}{N_0^2 \theta W} \log_2 e \right\}
\end{aligned}$$

■

## 2) Path Delays Unknown to Receiver:

*Theorem 1: Part 2:* DSSS systems with duty cycle achieve  $C_{\text{DSSS}} \rightarrow C_{\text{AWGN}}$  as  $W \rightarrow \infty$  if  $\frac{L \log W}{W} \rightarrow 0$ .

*Proof:* The proof is based on Proposition 3 and equation (5) that relates the mutual information in the case of channel knowledge of the path delays with the mutual information in the general case. With no channel knowledge at the receiver (path delays and gains unknown), we get:

$$\begin{aligned}
I(X;Y) \text{ [b/s]} &\geq C_{\text{AWGN}} \\
&\quad - \min_{0 < \theta \leq 1} \left\{ \frac{\theta L}{T_c} \log_2 \left( 1 + \frac{P}{N_0} \frac{T_c}{\theta L} \right) \right. \\
&\quad \left. + \frac{3P^2}{N_0^2 \theta W} \log_2 e + \frac{\theta L}{T_c} \log_2 (W T_d) \right\}
\end{aligned} \tag{10}$$

The third penalty term describes the penalty due to path delays, from (5). This term is a bound on the penalty, that depends linearly on the number of path delays per unit time.

At the limit of infinite bandwidth the bound equals the AWGN capacity if

- $\theta W \rightarrow \infty$
- $\theta L \rightarrow 0$
- $\theta L \log W \rightarrow 0$

The second condition may be dropped, as the third is stronger. These conditions can be met so simultaneously if  $L \log W$  is sub-linear in  $W$ , that is if  $\frac{L \log W}{W} \rightarrow 0$ . ■

### B. When is the Channel Capacity Not Achieved?

An additional assumption on gains is used in this section: the gains are uniformly upper bounded by  $|G_l| \leq \frac{B}{\sqrt{L}}$ , this is a technical condition that follows [3].

*Converse to Theorem 1:* DSSS systems with duty cycle where the path gains are unknown to the receiver and uniformly bounded by  $|G_l| \leq \frac{B}{\sqrt{L}}$  with a constant  $B$ , achieve  $C_{\text{DSSS}} < C_{\text{AWGN}}$  in the limit  $W \rightarrow \infty$  if  $\frac{L}{W} \rightarrow \alpha$  and  $\alpha > 0$ . This bound holds whether the receiver knows the path delays or it does not.

*Proof:* We first note that the mutual information in a system where the receiver knows the path delays upper bounds the mutual information in the general case:

$$I(X; Y) = I(X; Y|D) - I(D; X|Y) \leq I(X; Y|D)$$

So we only need to prove the theorem regarding the conditional mutual information, where the receiver knows the path delays.

The proof is based on the following upper bound on the mutual information.

*Proposition 4:* DSSS systems with duty cycle parameter  $\theta$  achieve

$$I(X; Y|D) \text{ [b/s]} \leq W\theta \log_2 \left( 1 + \frac{P}{N_0 W \theta} \right) \quad (11)$$

If the duty cycle is chosen so that  $\theta L \rightarrow \infty$  as  $W \rightarrow \infty$ , then a second upper bound holds in the limit:

$$I(X; Y|D) \text{ [b/s]} \leq C_{\text{AWGN}} \frac{2T_d B^2 d}{T_c} \quad (12)$$

$d$  is defined in (4) for pseudo-random chips. For IID chips, the bound holds with  $d = 1$ .

*Discussion of Proposition 4:* We now look at two different possibilities regarding the duty cycle parameter  $\theta$ :

- $\theta W < \infty$ . In this case the bound (11) is strictly lower than  $C_{\text{AWGN}}$ .
- $\theta W \rightarrow \infty$  (as  $W \rightarrow \infty$ ). Using our assumption on the number of channel paths we get  $\theta L \rightarrow \infty$ , so the second bound (12) becomes relevant. In situations where  $T_d \ll T_c$ , this bound is significantly lower than the AWGN capacity.

If the number of paths is sub-linear in  $W$ , the duty cycle can be chosen so that  $\theta W \rightarrow \infty$  and  $\theta L \rightarrow 0$  and the bounds in (12) become irrelevant. In this case the upper bound converges to  $C_{\text{AWGN}}$  in the limit of infinite bandwidth.

To summarize the behavior of the bound in the limit, in the case of a linear increase of the number of paths with the bandwidth, the upper bound is lower than the AWGN capacity in normal operating conditions ( $T_d \ll T_c$ ). The upper bound equals  $C_{\text{AWGN}}$  in the case of a sub-linear increase of the number of paths with the bandwidth.

*Proof of Proposition 4* We start with a simple bound:

$$I(X; Y|D) \text{ [b/s]} \leq W\theta \log_2 \left( 1 + \frac{P}{N_0 W \theta} \right)$$

This is the capacity of an AWGN channel used with duty cycle  $\theta$ . It upper bounds the data rate for systems with channel knowledge at the receiver. In order to achieve the capacity at the limit of infinite bandwidth, the duty cycle parameter must be chosen so that  $\theta W \rightarrow \infty$ .

To prove (12) we follow Theorem 2 of [3], that gives an upper bound on the mutual information.

$$\begin{aligned} I(X; Y|D) &\leq E_G \log \left( E_H \exp \left[ 2\mathcal{E} \text{Re} \left\{ \sum_{l=1}^L H_l G_l \right\} \right] \right) \\ &+ 2\mathcal{E} \frac{B^2}{L} E_D \left[ \sum_{l=1}^L \sum_{m=1}^L E_X |C(D_l, D_m) - \delta_{lm}| \right] \log_2 e \end{aligned} \quad (13)$$

where  $G$  and  $H$  are identically distributed and independent,  $|H_l| \leq \frac{B}{\sqrt{L}}$ .

The first part of (13):

$$\begin{aligned} &\log E_H \exp \left[ 2\mathcal{E} \text{Re} \left\{ \sum_{l=1}^L H_l G_l \right\} \right] \\ &= \log E_{|H|} E_{\psi} \exp \left[ 2\mathcal{E} \text{Re} \left\{ \sum_{l=1}^L e^{-j\psi_l} |H_l| G_l \right\} \right] \end{aligned}$$

The phases  $\{\psi_l\}$  equal 0 or  $\pi$  with probability 1/2.

$$E_{\psi} \exp(\operatorname{Re}(ae^{j\psi_l})) = \frac{e^a + e^{-a}}{2} \approx 1 + \frac{a^2}{2} \quad \text{for } a \ll 1$$

$$\begin{aligned} E_{|H|} E_{\psi} \exp \left[ 2\mathcal{E} \operatorname{Re} \left\{ \sum_{l=1}^L e^{-j\psi_l} |H_l| G_l \right\} \right] \\ \leq E_{|H|} \prod_{l=1}^L (1 + 2\mathcal{E}^2 |H_l|^2 |G_l|^2) \end{aligned}$$

and the condition is  $\frac{2\mathcal{E}B^2}{L} \ll 1$ , which holds if  $\theta L \rightarrow \infty$ .

$$\begin{aligned} E_G \log \left( E_H \exp \left[ 2\mathcal{E} \operatorname{Re} \left\{ \sum_{l=1}^L H_l G_l \right\} \right] \right) \\ \leq E_G \log E_{|H|} \prod_{l=1}^L (1 + 2\mathcal{E}^2 |H_l|^2 |G_l|^2) \end{aligned}$$

Using Jensen's inequality:

$$\begin{aligned} E_G \log \left( E_H \exp \left[ 2\mathcal{E} \operatorname{Re} \left\{ \sum_{l=1}^L H_l G_l \right\} \right] \right) \\ \leq \sum_{l=1}^L \log (1 + E_{|G_l|, |H_l|} 2\mathcal{E}^2 |H_l|^2 |G_l|^2) \\ = \sum_{l=1}^L \log \left( 1 + \frac{2\mathcal{E}^2}{L^2} \right) \\ = L \log \left( 1 + \frac{2\mathcal{E}^2}{L^2} \right) \end{aligned}$$

with  $\mathcal{E} = \frac{2PT_c}{N_0\theta}$ :

$$\begin{aligned} E_G \log \left( E_H \exp \left[ 2\mathcal{E} \operatorname{Re} \left\{ \sum_{l=1}^L H_l G_l \right\} \right] \right) \\ \leq L \log \left( 1 + \frac{8P^2 T_c^2}{N_0^2 \theta^2 L^2} \right) \\ \leq \frac{8P^2 T_c^2}{N_0^2 \theta^2 L} \log e \end{aligned}$$

The second part of (13):

- For IID chips: Using (3)

$$\begin{aligned}
& 2\mathcal{E} \frac{B^2}{L} E_D \left[ \sum_{l=1}^L \sum_{m=1}^L E_X |C(D_l, D_m) - \delta_{lm}| \right] \log_2 e \\
& \leq 2\mathcal{E} \frac{B^2}{L} E_D \left( \frac{3L}{\sqrt{K_c}} + \frac{L^2 - L}{K_c} \right) \log_2 e \\
& = 2\mathcal{E} B^2 \left( \frac{3}{\sqrt{K_c}} + \frac{L-1}{K_c} \right) \log_2 e \\
& \approx \frac{2\mathcal{E} B^2 L}{K_c} \log_2 e \quad \text{for } L \gg 1 \\
& \leq \frac{4PT_d}{N_0\theta} B^2 \log_2 e
\end{aligned}$$

The last inequality follows from  $L \leq T_d W$ .

- For pseudo-random chips:

$$\begin{aligned}
& 2\mathcal{E} \frac{B^2}{L} E_D \left[ \sum_{l=1}^L \sum_{m=1}^L E_X |C(D_l, D_m) - \delta_{lm}| \right] \log_2 e \\
& \leq 2\mathcal{E} B^2 L \frac{d}{K_c} \log_2 e \\
& = \frac{4PL}{N_0\theta W} B^2 d \log_2 e \\
& \leq \frac{4PT_d}{N_0\theta} B^2 d \log_2 e
\end{aligned} \tag{14}$$

so (14) is valid in both cases of input signals, and for IID chips we take  $d = 1$ .

Putting the two parts back into (13):

$$I(X; Y|D) \leq \frac{8P^2 T_c^2}{N_0^2 \theta^2 L} \log e + \frac{4PT_d}{N_0\theta} B^2 d \log_2 e$$

In units of [bits/sec]:

$$I(X; Y|D) \text{ [b/s]} \leq \frac{8P^2 T_c}{N_0^2 \theta L} \log_2 e + \frac{4PT_d}{N_0 T_c} B^2 d \log_2 e \tag{15}$$

Using  $\theta L \rightarrow \infty$  we get

$$I(X; Y|D) \text{ [b/s]} \leq \frac{4PT_d}{N_0 T_c} B^2 d \log_2 e = C_{\text{AWGN}} \frac{2T_d B^2 d}{T_c}$$

■

## VII. PPM BOUNDS

### A. When is the Channel Capacity Achieved? (Path Delays Known to Receiver)

*Theorem 2:* PPM systems with duty cycle, where the receiver knows the path delays, achieve  $C_{\text{PPM}} \rightarrow C_{\text{AWGN}}$  as  $W \rightarrow \infty$  if  $\frac{L}{\log W} \rightarrow 0$  and the path gains  $\{G_l\}_{l=1}^L$  satisfy (A)  $\max_{1 \leq i \leq L} |G_i| \rightarrow 0$  in probability as  $L \rightarrow \infty$ , and (B)  $E_G = \sum_{i=1}^L G_i^2 \rightarrow 1$  in probability as  $L \rightarrow \infty$ . Note that if the gains are Gaussian then the above conditions hold.

*Proof:* We start by breaking the mutual information in two parts:

$$I(X; Y|D) \geq I(X; Y|G, D) - I(Y; G|X, D) \quad (16)$$

The maximal data rate achievable by systems that use the PPM signals defined in Section II-B is the maximum of (16) over the duty cycle parameter  $\theta$ .

$$\begin{aligned} C_{\text{PPM}} &= \max_{\theta} I(X; Y|D) \\ &\geq \max_{\theta} [I(X; Y|G, D) - I(Y; G|X, D)] \end{aligned} \quad (17)$$

The first part of (17) describes the throughput of a system that knows the channel perfectly. Section VII-A.2 shows that it approaches  $C_{\text{AWGN}}$  if the duty cycle parameter  $\theta$  is chosen appropriately. This result is shown by demonstrating that the probability of error diminishes as the bandwidth increases while the data rate is as close to  $C_{\text{AWGN}}$  as desired. Section VII-A.1 calculates the penalty on the system for its channel gain knowledge and shows that it diminishes for our choice of duty cycle parameter  $\theta$ .

The receiver we use in the analysis of probability of error is based on a matched filter, it is derived from the optimal (maximum likelihood) receiver in the case of a channel with a single path. PPM signals are composed of *orthogonal* symbols. When a PPM signal is transmitted over an impulsive (single path) channel, the orthogonality of the symbols is maintained at the receiver side.

Considering a multipath channel, the received symbol values are no longer orthogonal, so the matched filter receiver is no longer optimal. The non-orthogonality of the received symbols has an adverse effect on receiver performance. As the number of channel paths increases, the received symbols become increasingly non-orthogonal, and the receiver performance is degraded. The matched filter receiver can sustain a growth of the number of channel paths (as the bandwidth

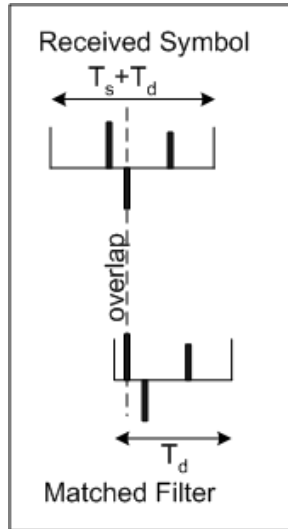


Fig. 7. A received symbol and the matched filter showing a partial overlap.

increases), but this growth must not be too rapid. To put it more formally, our system achieves the channel capacity in the limit of infinite bandwidth, if the number of paths obeys  $\frac{L^6}{W} \rightarrow 0$ .

For each possible transmitted symbol value, the receiver matches the signal with a filter that has  $L$  fingers at the right delays (Figure 7). The receiver uses a threshold parameter  $A = \alpha\sqrt{\mathcal{E}/N}$  for deciding on the transmitted value. If one output only is above  $A$ , the input is guessed according to this output. If none of the outputs pass  $A$ , or there are two or more that do, an error is declared.

We calculate an upper bound on the probability of error of this system, and show that it converges to zero as the bandwidth increases, if the number of channel paths does not increase too rapidly, namely  $\frac{L^6}{W} \rightarrow 0$ , and the duty cycle parameter is chosen properly.

The second part of (17) describes a penalty due to unknown path gains, it is analyzed separately in Section VII-A.1, the upper bound calculated there does not depend on the coding used by the transmitter, and it diminishes as the bandwidth increases for our choice of duty cycle parameter.

We summarize here the conditions for convergence of (17), to bring the conclusion of the following lengthy calculation: The system uses duty cycle with parameter  $\theta$  over coherence periods; The first part of (17) converges to  $C_{\text{AWGN}}$  in the limit of infinite bandwidth, if the following conditions take place (end of Section VII-A.2):

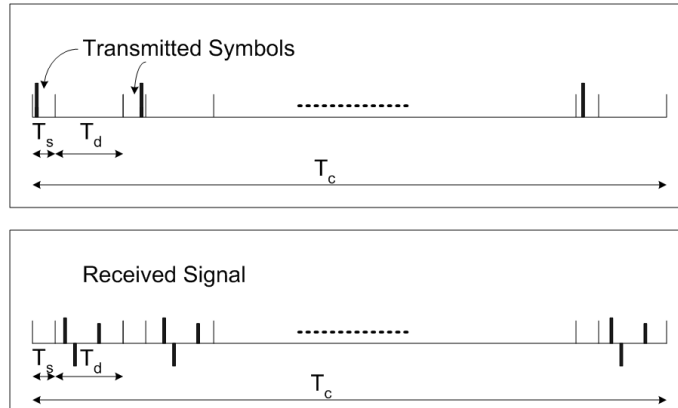


Fig. 8. The transmitted PPM signal, with guard time of  $T_d$  between symbols, and the received signal.

- $\theta \log L \rightarrow 0$
- $\frac{L^6}{W} \rightarrow 0$
- $\theta \log W \sim \text{const}$

The second part of (17) contains the penalty for channel (gain) uncertainty; it converges to zero if  $\theta L \rightarrow 0$  (Section VII-A.1). These conditions can exist simultaneously if  $\frac{L}{\log W} \rightarrow 0$ .

1) *Upper Bound on  $I(Y; G|X, D)$* : The position of the signal fingers is known as  $X$  and  $D$  are known.

$$I(Y; G|X, D) = h(Y|X, D) - h(Y|X, G, D) \quad (18)$$

The first part of (18) is upper bounded by the differential entropy in the case of Gaussian path gains. Considering the signals during a coherence period ( $N$  transmitted symbols), the discretized received signal is composed of  $NM_r$  values (chips), where  $M_r = W(T_s + T_d)$ . Given  $X$  and  $D$ , it is known which chips contain signal and which contain only noise (Figure 8). The  $NM_r$  received values are distributed as follows:

- $N(M_r - L)$  are IID Gaussian  $\sim N(0, 1)$ .
- $NL$  values are divided into groups of size  $N$ . Each group is independent of the other groups, it has zero mean and its correlation matrix is

$$\Lambda = \frac{2P(T_s + T_d)}{\theta N_0 L} + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

The differential entropy (in bits per coherence time) is bounded by

$$h(Y|X, D) \leq \frac{N(M_r - L)}{2} \log_2(2\pi e) + \frac{L}{2} \log_2((2\pi e)^N |\Lambda|)$$

The determinant  $|\Lambda|$  is the product of the eigenvalues of  $\Lambda$ : 1 with multiplicity  $N - 1$  and  $1 + \frac{\mathcal{E}}{L}$  with multiplicity one

$$|\Lambda| = 1 + \frac{\mathcal{E}}{L}$$

The second part of (18) is given by

$$h(Y|X, D, G) = \frac{NM_r}{2} \log_2(2\pi e)$$

Combining both parts, and translating to units of [bits/sec]:

$$\begin{aligned} I(Y; G|X, D) \text{ [b/s]} &\leq \frac{\theta L}{2T_c} \log_2 |\Lambda| \\ &= \frac{\theta L}{2T_c} \log_2 \left( 1 + \frac{\mathcal{E}}{L} \right) \\ &= \frac{\theta L}{2T_c} \log_2 \left( 1 + \frac{2P(T_s + T_d)N}{\theta N_0 L} \right) \end{aligned} \quad (19)$$

The bound (19) converges to zero as  $\theta L \rightarrow 0$ .

2) *Lower Bound on  $\max_{\theta} I(Y; X|G, D)$* : This bound holds if the path gains  $\{G_l\}_{l=1}^L$  satisfy (A)  $\max_{1 \leq i \leq L} |G_i| \rightarrow 0$  in probability as  $L \rightarrow \infty$ , and (B)  $E_G = \sum_{i=1}^L G_i^2 \rightarrow 1$  in probability as  $L \rightarrow \infty$ .

We first show that the Gaussian distribution  $G_l \sim N(0, 1/L)$  satisfies these condition. Condition (B) follows easily from the law of large numbers. To prove that condition (A) holds, we use the following well-known tail estimate for a standard normal  $Z$  and any  $x > 0$ :

$$P(Z > x) \leq \frac{1}{\sqrt{2\pi}x} \exp(-x^2/2) \quad (20)$$

Using  $\beta = 2\sqrt{\frac{\log L}{L}}$  we get

$$\begin{aligned} P\left(\max_{1 \leq l \leq L} |G_l| > \beta\right) &\leq LP(|G_1| > \beta) \\ &\leq LP(|Z| > \beta\sqrt{L}) \\ &\leq \frac{L\sqrt{L}}{\sqrt{2\pi \log L}} \exp(-2 \log L) \\ &\rightarrow 0 \quad \text{as } L \rightarrow \infty \end{aligned}$$

Clearly  $\beta \rightarrow 0$  as  $L \rightarrow \infty$ .

*Analysis of the Signals in the Receiver:* For every symbol, the receiver calculates

$$s_i = \sum_{j=1}^L G_j Y_{i+D_j-1} \quad i = 1, \dots, WT_s$$

Assuming that  $x_1$  was transmitted the desired output is Gaussian with

$$\begin{aligned} E[s_1] &= \sqrt{\mathcal{E}/N} E_G \\ \sigma_{s_1}^2 &= E_G \end{aligned}$$

There are up to  $L^2 - L$  Gaussian overlap terms that contain part of the signal (Figure 7). Each of these overlap terms can be described by a set OL of pairs of path gains, each index (between 1 and  $L$ ) indicating a path in the channel response.

$$\text{OL} = \{(I_{1,1}, I_{1,2}), \dots, (I_{|\text{OL}|,1}, I_{|\text{OL}|,2})\}$$

The number of overlap terms may vary in the range  $1 \leq |\text{OL}| \leq L - 1$  and the indices take valued between 1 and  $\lfloor T_d W \rfloor$ . The pairs of indices are composed of two different indices, because the case where the filter position corresponds to the actual signal position is already accounted for in  $s_1$ .

Given the overlap, the overlap terms  $s_{\text{overlap}|\text{OL}}$  are Gaussian with

$$\begin{aligned} E[s_{\text{overlap}|\text{OL}}] &= \sum_{i=1}^{|\text{OL}|} G_{I_{i,1}} G_{I_{i,2}} \sqrt{\mathcal{E}/N} \\ E[s_{\text{overlap}|\text{OL}}^2] &\leq \sum_{i=1}^{|\text{OL}|} G_{I_{i,1}}^2 G_{I_{i,2}}^2 \mathcal{E}/N + E_G \end{aligned}$$

Assuming a small number of paths, the probability that there are two or more overlaps converges to zero as the bandwidth increases to infinity, see the proof of this convergence in Section VII-A.3. A suitable assumption on the number of paths is  $\frac{L^4}{W} \rightarrow 0$ .

In addition, there are up to  $WT_s - 1$  Gaussian noise terms:

$$\begin{aligned} E[s_{\text{noise}}] &= 0 \\ E[s_{\text{noise}}^2] &= E_G \end{aligned}$$

For each possible transmitted symbol value, the receiver compares  $\{s_i\}_{i=1}^{WT_s}$  to a threshold  $A = \alpha \sqrt{\mathcal{E}/N} = \alpha \sqrt{\frac{2PT_c}{N_0 \theta N}}$  where  $\alpha \in (0, 1)$ . If one output only is above  $A$ , the input is guessed according to this output. If none of the outputs pass  $A$ , or there are two or more that do, an

error is declared.

There are three types of error events, and the error probability is upper bounded using the union bound:

$$\begin{aligned}
P(\text{error}) &\leq P(s_1 \leq A) + (L^2 - L)P(s_{\text{overlap}} \geq A) \\
&\quad + (WT_s - 1)P(s_{\text{noise}} \geq A) \\
&\leq P(s_1 \leq A) + L^2P(s_{\text{overlap}} \geq A) \\
&\quad + WT_sP(s_{\text{noise}} \geq A)
\end{aligned} \tag{21}$$

The first probability is bounded using the Chebyshev inequality, and the second and third using the normal tail estimate.

*First Error Event:* Recall  $s_1$  has expectation  $\sqrt{\mathcal{E}/N}E_G$  and variance  $E_G$ . From the Chebyshev inequality,

$$\begin{aligned}
P(s_1 \leq A) &\leq \frac{\sigma_{s_1}^2}{(E[s_1] - A)^2} \\
&= \frac{E_G}{(E_G - \alpha)^2 \mathcal{E}/N}
\end{aligned}$$

Since  $\alpha < 1$  and  $E_G \rightarrow 1$  in probability, for large  $L$  the ratio  $E_G/(E_G - \alpha)^2$  is bounded. Since  $\mathcal{E}/N \rightarrow \infty$ , the probability converges to 0.

*Second Error Event:* The probability that an overlap term exceeds the threshold is expressed as a sum over the  $L - 1$  possibilities of the number of overlap positions:

$$\begin{aligned}
&L^2P(s_{\text{overlap}} \geq A) \\
&= L^2 \sum_{i=1}^{L-1} P(|\text{OL}| = i) P(s_{\text{overlap}} \geq A | |\text{OL}| = i)
\end{aligned}$$

Section VII-A.3 shows that if the number of paths is such that  $\frac{L^4}{W} \rightarrow 0$ , then the probability of overlap at more than one position diminishes as  $W \rightarrow \infty$ .

$$P(|\text{OL}| > 1) \rightarrow 0$$

In order to ensure that the overlap terms with more than one overlap position are insignificant in the calculation of the probability of error, we require  $\frac{L^6}{W} \rightarrow 0$ , and then get in the limit of

large bandwidth

$$L^2 P(s_{\text{overlap}} \geq A) \rightarrow L^2 P(s_{\text{overlap}} \geq A | |\text{OL}| = 1)$$

The condition  $|\text{OL}| = 1$  is omitted in the remainder of the calculation.

Recall that in the single overlap case  $s_{\text{overlap}}$  is normal with mean  $\mu = G_l G_m \sqrt{\mathcal{E}/N}$  with  $l \neq m$  and variance  $E_G = \sum G_i^2$ . Hence  $(s_{\text{overlap}} - \mu)/\sqrt{E_G}$  is a standard normal.

By assumption,  $\max |G_i| \rightarrow 0$  and  $E_G \rightarrow 1$  in probability, so for  $L$  large we can assume  $\mu \leq \alpha/2 \sqrt{\mathcal{E}/N} = A/2$  and  $E_G \leq 4$ . Then

$$\begin{aligned} P(s_{\text{overlap}} \geq A) &= P((s_{\text{overlap}} - \mu)/\sqrt{E_G} \geq (A - \mu)/\sqrt{E_G}) \\ &\leq P((s_{\text{overlap}} - \mu)/\sqrt{E_G} \geq A/4) \end{aligned}$$

Using the normal tail estimate (20), we obtain

$$\begin{aligned} L^2 P(s_{\text{overlap}} \geq A) &\leq L^2 P(Z \geq A/4) \\ &\leq \exp(2 \ln L - A^2/32 - \ln A/2) \end{aligned}$$

where  $Z$  stands for a standard normal. In order to ensure convergence to zero of this probability, it is enough to have  $(\ln L)/A^2 \rightarrow 0$ . Recalling  $A = \alpha \sqrt{\mathcal{E}/N} = \alpha \sqrt{\frac{2PT_c}{N_0 \theta N}}$ , we get an equivalent condition:  $\theta \ln L \rightarrow 0$ .

*Third Error Event:* Recall  $s_{\text{noise}}$  is normal with mean 0 and variance  $E_G$ . The third probability in (21) is upper bounded using the normal tail estimate (20) for the standard normal  $Z = s_{\text{noise}}/\sqrt{E_G}$ :

$$P(s_{\text{noise}} \geq A) \leq (2\pi)^{-1/2} (\sqrt{E_G}/A) \exp(-A^2/(2E_G))$$

$$WT_s P(s_{\text{noise}} \geq A) \leq \exp[\ln(WT_s) - A^2/(2E_G) - \ln A]$$

The data rate (in bits/sec) is

$$R = \frac{\theta N}{T_c} \log_2(WT_s) = \frac{\theta N}{T_c} \log_2 e \ln(WT_s)$$

and the capacity  $C_{\text{AWGN}} = P/N_0 \log_2 e$ , so

$$(C_{\text{AWGN}}/R) \ln(WT_s) = (PT_c)/(\theta N N_0) = \mathcal{E}/2N$$

Since  $A^2 = \alpha^2 \mathcal{E}/N$ , we obtain the bound

$$WT_s P(s_{\text{noise}} \geq A) \leq \exp \left[ \ln(WT_s) - (\alpha^2/E_G)(C_{AWGN}/R) \ln(WT_s) - \ln \frac{A}{\sqrt{E_G}} \right]$$

Since  $E_G \rightarrow 1$ , the bound converges to 0 as long as  $\alpha^2 > R/C_{AWGN}$ . This can be achieved for any data rate below the AWGN capacity.

*Achieving Capacity:* So far we have assumed  $\alpha$  is a constant smaller than 1, so the communication system can achieve any rate below  $C_{AWGN}$ . The duty cycle parameter  $\theta$  must vary as  $\frac{1}{\log W}$ . To achieve asymptotically  $C_{AWGN}$ , the parameter  $\alpha$  must approach 1 as the bandwidth increases, and the following conditions need to be satisfied:

- $(E_G - \alpha)^2 \log W \rightarrow \infty$  in probability (first error event)
- $(\alpha^2/E_G)(C_{AWGN}/R) \geq 1$  with probability  $\rightarrow 1$  (third error event)

The exact choice of  $\alpha$  depends on the rate at which  $E_G$  converges to 1.

*Summary of the Bound:* The system uses IID symbols, a duty cycle  $\theta$  and a threshold  $A = \alpha \sqrt{\mathcal{E}/N} = \alpha \sqrt{\frac{2PT_c}{N_0 \theta N}}$  where  $\alpha \in (0, 1)$ .

We calculated an upper bound on the error probability

$$P(\text{error}) \leq \text{upper bound}(W, L, \frac{P}{N_0}, \alpha, \theta)$$

that converges to zero as  $W \rightarrow \infty$  if

- $\frac{L^6}{W} \rightarrow 0$  (second error event)
- $\theta \log W \sim \text{const}$  (to ensure positive rate  $R$ )
- $\theta \log L \rightarrow 0$  (second error event)
- $\theta L \rightarrow 0$  (penalty for unknown gains)

If  $\frac{L}{\log W} \rightarrow 0$  these the conditions can be realized simultaneously, namely it is possible to choose a duty cycle parameter  $\theta$  that satisfies all the conditions.

Note that if the path delays are not known, the additional penalty (5) increases as  $\theta L \log W$ , which diverges, so the above proof is not useful.

3) *Estimation of Number of Overlap Terms:* The number of possible path positions is  $L_m = \lfloor WT_d \rfloor$ . We assume that, over one coherence period, the  $L$  delays are chosen uniformly at random among

the  $\binom{L_m}{L}$  possibilities. We prove that if the number of paths grows slowly enough, then with probability converging to 1 there will be at most one overlap between the set of delays and any of its translations.

*Definition 1:* For any set  $S \subset \mathbf{Z}$  and any integer  $t \in \mathbf{Z}$ , we denote by  $S + t$  the translation of  $S$  by  $t$ :  $S + t = \{s + t | s \in S\}$ .

$S$  corresponds to the received symbol when  $x_1$  is transmitted, and  $S + t$  corresponds to  $x_{t+1}$ .

For integers  $1 \leq L \leq L_m$  pick a random subset  $S \subset \{1, \dots, L_m\}$  uniformly among all subsets of  $\{1, \dots, L_m\}$  with  $L$  elements. Let  $\mathbf{P}_{L_m, L}$  be the law of  $S$ ; when there is no ambiguity we drop the subscripts and refer to it as  $\mathbf{P}$ .

*Theorem 5:* Assume  $L^4/L_m \rightarrow 0$  as  $L_m \rightarrow \infty$ , and a set  $S$  is chosen according to  $\mathbf{P}_{L_m, L}$ . Then  $\mathbf{P}_{L_m, L}(|S \cap (S + t)| > 1 \text{ for some } t \neq 0) \rightarrow 0$ .

Note that  $t$  can take both positive and negative values. We emphasize that the theorem says that with high probability, **none** of the translates will have more than one overlap. The proof requires the following

*Lemma 6:* Fix  $t \neq 0$ , and let  $A$  be a set such that  $A \cup (A - t) \subset \{1, \dots, L_m\}$ , and  $|A| \leq L$ . Then

$$\mathbf{P}(A \subset S \cap (S + t)) = [L]_a / [L_m]_a \leq (L/L_m)^a \quad (22)$$

where  $a = |A \cup (A - t)|$  and  $[x]_a = x(x-1)\dots(x-a+1)$ .

*Proof:* Clearly  $A \subset S \cap (S + t)$  is equivalent to  $A \subset S$ ,  $(A - t) \subset S$ , hence  $S$  has to contain  $A \cup (A - t)$ . Hence  $a$  elements of  $S$  are fixed, while the remaining  $L - a$  ones can be chosen in  $\binom{L_m - a}{L - a}$  ways. The total number of subsets of  $\{1, \dots, L_m\}$  with  $L$  elements is  $\binom{L_m}{L}$ , hence

$$\mathbf{P}(A \subset S \cap (S + t)) = \binom{L_m - a}{L - a} / \binom{L_m}{L} = [L]_a / [L_m]_a$$

The inequality (22) follows easily. ■

**Note.** If  $A \cup (A - t)$  is not a subset of  $\{1, \dots, L_m\}$ , or if  $|A| > L$ , then the probability (22) is 0.

We are ready to obtain estimates. Fix  $t > 0$ . If  $|S \cap (S + t)| \geq 2$ , then  $S \cap (S + t)$  must contain  $A = \{i, j\}$  for some  $t + 1 \leq i < j \leq L_m$ . There are  $\binom{L_m - t}{2}$  such sets  $A$ . Exactly  $L_m - 2t$  of them (namely  $\{i, i + t\}$  for  $t + 1 \leq i \leq L_m - t$ ) have  $|A \cup (A - t)| = 3$ ; all the others have

$|A \cup (A - t)| = 4$ . Hence

$$\begin{aligned}
& \mathbf{P}(|S \cap (S + t)| \geq 2) \\
& \leq \sum_{t+1 \leq i < j \leq L_m} \mathbf{P}(\{i, j\} \subset S \cap (S + t)) \\
& \leq (L_m - 2t)(L/L_m)^3 + \binom{L_m - t}{2} (L/L_m)^4 \\
& \leq \frac{1}{L_m^2} (L^3 + L^4)
\end{aligned}$$

The same estimate holds for  $t < 0$ . Hence

$$\begin{aligned}
& \mathbf{P}(|S \cap (S + t)| > 1 \text{ for some } t \neq 0) \\
& \leq \sum_{\substack{-L_m \leq t \leq L_m \\ t \neq 0}} \mathbf{P}(|S \cap (S + t)| > 1) \\
& \leq 2L_m \frac{1}{L_m^2} (L^3 + L^4) \\
& \leq 4L^4/L_m
\end{aligned}$$

and the proof of Theorems 5 and 2 is complete. ■

### B. When is the Channel Capacity Not Achieved?

*Converse to Theorem 2:* PPM systems with a lower bounded symbol time transmitting over a channel with Gaussian path gains that are unknown to the receiver, achieve  $C_{\text{PPM}} \rightarrow 0$  as  $W \rightarrow \infty$  if  $\frac{L}{\log W} \rightarrow \infty$ . This result holds whether the receiver knows the path delays or it does not.

The signals we consider are PPM with symbol time that may depend on the bandwidth, but cannot exceed the coherence period of the channel and cannot diminish (by assumption). The symbol time is divided into positions separated by  $\frac{1}{W}$ . Guard time may be used, no restriction is imposed over it, we use  $T_{\text{sympb}}$  to denote the the overall symbol time, that includes the guard

time. Each symbol is of the form:

$$X_i = \begin{cases} \sqrt{\frac{WT_{\text{syimb}}}{\theta}} & \text{one position of each group of } \lfloor T_{\text{syimb}}W \rfloor \\ & \text{with } n\lfloor T_{\text{syimb}}W \rfloor \leq i \leq n\lfloor T_{\text{syimb}}W \rfloor + \lfloor T_{\text{syimb}}W \rfloor - 1 \\ 0 & \text{other positions} \end{cases}$$

$$n = 0, 1, \dots, \left\lfloor \frac{T_c}{T_{\text{syimb}}} \right\rfloor - 1 \quad (\text{symbol counter})$$

$$i = 0, 1, \dots, \lfloor T_c W \rfloor - 1 \quad (\text{position counter})$$

The number of symbols transmitted over a single coherence period is  $N = \frac{T_c}{T_{\text{syimb}}}$ . We assume that  $N$  is a whole number, this assumption does not alter the result we prove here. Duty cycle or any other form of infrequent transmission may be used over any time period. We analyze systems that use duty cycle over coherence periods, because this choice yields the highest data rate that serves as an upper bound. The channel is composed of  $L$  paths with independent and identically distributed Gaussian gains, and delays in the range  $[0, T_d)$ .

Edge effects between coherence periods are not considered, they may add a complication to the analysis, without contributing to the understanding of the problem or the solution.

*Outline of the Proof of The Converse to Theorem 2:* The mutual information of the transmitted and received signals is upper bounded by the mutual information when the receiver knows the path delays. This, in turn, is upper bounded in two ways: the first is the PPM transmitted bit rate for a system that does not use coding, and the second is based on the performance of a simple PPM system with no inter-symbol interference. The proof is based on the conditions where the upper bound we calculate on the mutual information converges to zero as the bandwidth increases.

*Proof:* We first point out that the mutual information of a system can only increase if the receiver is given information on the path delays:

$$I(X; Y) \leq I(X; Y|D)$$

We calculate an upper bound on PPM mutual information with a real Gaussian multipath channel, in [bits/sec]:

*Proposition 7:*

$$I(X; Y|D) \text{ [b/s]} \leq \max_{0 < \theta \leq 1} \min \{I_1(\theta), I_2(\theta)\} \quad (23)$$

$$I_1(\theta) \text{ [b/s]} \equiv \frac{\theta \log_2(W T_{\text{symp}})}{T_{\text{symp}}}$$

$$I_2(\theta) \text{ [b/s]} \equiv \frac{\theta}{2T_s} \sum_{i=1}^{W(T_d+T_{\text{symp}})} \log_2 \left( 1 + p_i \frac{2PT_{\text{symp}}}{\theta N_0 L} \right) \quad (24)$$

$$-\frac{\theta L}{2T_c} \log_2 \left( 1 + \frac{2PT_c}{\theta N_0 L} \right) \quad (25)$$

where  $p_1, \dots, p_{W(T_d+T_{\text{symp}})}$  satisfy  $0 \leq p_i \leq 1$  and  $\sum_{i=1}^{W(T_d+T_{\text{symp}})} p_i = L$ .

*Discussion of Proposition 7:* The first part of the bound,  $I_1(\theta)$ , is an upper bound on the the PPM bit rate for an uncoded system, it is a trivial upper bound on the mutual information.  $\theta$  is the fraction of time used for transmission, and the bound (23) is maximized over the choice of its value. The second part,  $I_2(\theta)$  depends on the number of channel paths  $L$ .

Using Proposition 7, the converse to Theorem 2 follows simply: The bound (23) is positive in the limit  $W \rightarrow \infty$  if both its parts are positive. We note that the symbol time  $T_{\text{symp}}$  is lower bounded by a constant that does not depend on the bandwidth. The first part,  $I_1(\theta)$ , is positive if the parameter  $\theta$  is chosen so that  $\theta \log(W T_{\text{symp}}) > 0$ . The second part  $I_2(\theta)$  is positive in the limit of infinite bandwidth if  $\theta L < \infty$ . If the environment is such that  $\frac{L}{\log W} \rightarrow \infty$ , the two conditions involving  $\theta$  cannot be met simultaneously by any choice of fractional transmission parameter. In this case, the bound (23) is zero in the limit of infinite bandwidth.

*Proof of Proposition 7:* The first part of (23) follows simply from the fact that  $I_1(\theta)$  is an upper bound on the transmitted data rate. For any choice of fractional transmission parameter  $\theta$ :

$$I(X; Y|D) \text{ [b/s]} \leq I_1(\theta) \text{ [b/s]}$$

The second part of (23) is proven by comparing the mutual information of our system, with that of a hypothetical system that is easier to analyze. The conditional mutual information  $I(X; Y|D)$  is upper bounded using a hypothetical system that transmits the same symbols as the system we analyze, and receivers them without inter-symbol interference (ISI). This is possible, for example, by using many pairs of transmitters and receivers, each pair transmitting symbols with long silence periods between them. The transmitter–receiver pairs are located in such a way that each receiver can ‘hear’ only its designated transmitter. This hypothetical system operates over

a channel identical to the one of the original system. The difference between the original system and the hypothetical system is apparent in the number of different noise samples they face, the hypothetical system receives more noise, it processes  $T'_c = T_c \frac{T_{\text{syemb}} + T_d}{T_{\text{syemb}}}$  positions per coherence period. In spite of this difference, the hypothetical system can achieve higher data rates, its mutual information is an upper bound on the mutual information in the original system. We use  $\tilde{I}(X; Y|D)$  to indicate the conditional mutual information of this system.

$$I(X; Y|D) \text{ [b/s]} \leq \tilde{I}(X; Y|D) \text{ [b/s]}$$

We now prove that for any choice of  $\theta$

$$\tilde{I}(X; Y|D) \text{ [b/s]} \leq I_2(\theta) \text{ [b/s]}$$

Each received symbol in the no-ISI system is composed of  $W(T_{\text{syemb}} + T_d)$  chips,  $L$  of them corresponding to the channel paths. All output positions have additive Gaussian noise of variance 1. The mutual information is given by

$$\tilde{I}(X; Y|D) \text{ [b/s]} = \frac{\theta}{T_c} \left[ \tilde{H}(Y|D) - \tilde{H}(Y|X, D) \right] \quad (26)$$

We start with the first part of (26):

$$\begin{aligned} \tilde{H}(Y|D) &\leq \sum_{i=1}^{WT'_c} \tilde{H}(Y_i|D) \leq \sum_{i=1}^{WT'_c} \tilde{H}(Y_i) \\ Y_i &\sim \begin{cases} N\left(0, 1 + \frac{2PT_{\text{syemb}}}{\theta N_0 L}\right) & \text{prob } p_i \\ N(0, 1) & \text{prob } 1 - p_i \end{cases} \\ &i = 1, \dots, W(T_d + T_{\text{syemb}}) \end{aligned}$$

$p_i$ , the probability of receiving signal energy in the  $i^{\text{th}}$  position, depends on the distribution of transmitted symbols, but there are exactly  $L$  positions in the received symbol that contain a path, thus

$$\sum_{i=1}^{W(T_d + T_{\text{syemb}})} p_i = L$$

and each probability value satisfies  $0 \leq p_i \leq 1$ .

$$\begin{aligned} \sigma_{Y_i}^2 &= E[Y_i^2] = 1 - p_i + p_i \left( 1 + \frac{2PT_{\text{syemb}}}{\theta N_0 L} \right) \\ &= 1 + p_i \frac{2PT_{\text{syemb}}}{\theta N_0 L} \end{aligned}$$

$$\begin{aligned}
\tilde{H}(Y_i) &\leq \frac{1}{2} \log(2\pi e \sigma_{Y_i}^2) \\
&= \frac{1}{2} \log \left( 2\pi e \left( 1 + p_i \frac{2PT_{\text{symb}}}{\theta N_0 L} \right) \right) \\
\tilde{H}(Y|D) &\leq \sum_{i=1}^{WT'_c} \tilde{H}(Y_i) \\
&\leq \sum_{T_c/T_{\text{symb}} \text{ symbols}} \sum_{i=1}^{W(T_d+T_{\text{symb}})} \frac{1}{2} \log \left( 2\pi e \left( 1 + p_i \frac{2PT_{\text{symb}}}{\theta N_0 L} \right) \right) \\
&= \frac{WT'_c}{2} \log(2\pi e) + \sum_{\text{symbols}} \sum_{i=1}^{W(T_d+T_{\text{symb}})} \frac{1}{2} \log \left( 1 + p_i \frac{2PT_{\text{symb}}}{\theta N_0 L} \right) \\
&= \frac{WT'_c}{2} \log(2\pi e) + \frac{T_c}{2T_{\text{symb}}} \sum_{i=1}^{W(T_d+T_{\text{symb}})} \log \left( 1 + p_i \frac{2PT_{\text{symb}}}{\theta N_0 L} \right)
\end{aligned}$$

Now for the second part of (26). For  $N$  transmitted symbols, the  $WT'_c$  received values are distributed as follows, when the input  $X$  and the delays  $D$  are known:

- $WT'_c - NL$  positions are IID Gaussians  $\sim N(0, 1)$ . The receiver knows which positions contain only noise, and which have signal as well.
- The number of positions with some signal is  $NL$ . These values are divided into groups of size  $N$ , each corresponding to a single path. Each group (at known positions) is independent of the other groups and its distribution is  $\sim N(0, \Lambda)$  where

$$\begin{aligned}
\Lambda &= \frac{2PT_{\text{symb}}}{\theta N_0 L} + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \\
-\tilde{H}(Y|X, D) &= -\frac{WT'_c - NL}{2} \log(2\pi e) \\
&\quad -\frac{L}{2} \log \left( (2\pi e)^N |\Lambda| \right) \tag{27}
\end{aligned}$$

The determinant  $|\Lambda|$  is the product of the eigenvalues of  $\Lambda$ : 1 with multiplicity  $N - 1$  and  $\left( 1 + \frac{2PT_{\text{symb}}N}{\theta N_0 L} \right)$  with multiplicity one

$$|\Lambda| = 1 + \frac{2PT_{\text{symb}}N}{\theta N_0 L} = 1 + \frac{2PT_c}{\theta N_0 L}$$

Simple manipulation yields

$$\begin{aligned} -\tilde{H}(Y|X, D) &= -\frac{WT'_c}{2} \log(2\pi e) \\ &\quad -\frac{L}{2} \log\left(1 + \frac{2PT_c}{\theta N_0 L}\right) \end{aligned}$$

We note that this expression depends on  $\text{SNR}_{\text{est}}$ ; It is similar to expressions in [3], Section III.C.

Combining the two parts into (26) we get

$$\begin{aligned} I(X; Y|D) \text{ [b/s]} &\leq \tilde{I}(X; Y|D) \text{ [b/s]} \\ &= \left[ \tilde{H}(Y|D) - \tilde{H}(Y|X, D) \right] \frac{\theta}{T_c} \\ &\leq \left[ \frac{WT'_c}{2} \log(2\pi e) + \frac{T_c}{2T_{\text{symb}}} \sum_{i=1}^{W(T_d+T_{\text{symb}})} \log\left(1 + p_i \frac{2PT_{\text{symb}}}{\theta N_0 L}\right) \right. \\ &\quad \left. - \frac{WT'_c}{2} \log_2(2\pi e) - \frac{L}{2} \log_2\left(1 + \frac{2PT_c}{\theta N_0 L}\right) \right] \frac{\theta}{T_c} \\ &= \frac{\theta}{2T_{\text{symb}}} \sum_{i=1}^{W(T_d+T_{\text{symb}})} \log\left(1 + p_i \frac{2PT_{\text{symb}}}{\theta N_0 L}\right) \\ &\quad - \frac{\theta L}{2T_c} \log_2\left(1 + \frac{2PT_c}{\theta N_0 L}\right) \\ &= I_2(\theta) \end{aligned}$$

Note that in the infinite bandwidth limit, if  $\theta L \rightarrow \infty$  than also  $\theta W \rightarrow \infty$  and  $I_2(\theta)$  converges to zero:

$$I_2(\theta) \longrightarrow \frac{\theta}{2T_{\text{symb}}} \sum_{i=1}^{W(T_d+T_{\text{symb}})} p_i \frac{2PT_{\text{symb}}}{\theta N_0 L} \log_2 e - \frac{\theta L}{2T_c} \frac{2PT_c}{\theta N_0 L} \log_2 e = 0$$

## APPENDIX

This section shows that for IID Gaussians  $\{x_i\}_{i=-\infty}^{\infty} \sim N(0, \frac{1}{\theta})$ , with the empirical correlation defined by:

$$C(m, n) = \frac{\theta}{K_c} \sum_{i=0}^{K_c-1} x_{i-m} x_{i-n}$$

We have

$$\sum_{m=1}^L \sum_{n=1}^L E_x |C(m, n) - \delta_{mn}| \leq \frac{3L}{\sqrt{K_c}} + \frac{L^2 - L}{K_c} \quad (28)$$

We first look at  $C(m, n)$  for  $m \neq n$ , and use the following inequality, that holds for any random variable:

$$E |c| \leq \sqrt{E [c^2]}$$

in our case:

$$\begin{aligned} E |C(m, n)| &\leq \sqrt{E [C(m, n)^2]} \\ &= \sqrt{\frac{\theta^2}{K_c^2} E \left[ \left( \sum_{i=0}^{K_c} x_{i-m} x_{i-n} \right) \right.} \\ &\quad \left. \sqrt{\left( \sum_{j=0}^{K_c} x_{j-m} x_{j-n} \right)} \right]} \\ &= \frac{\theta}{K_c} \sqrt{E \left[ \sum_{i=0}^{K_c} x_{i-m}^2 x_{i-n}^2 \right]} \\ &= \frac{1}{K_c} \end{aligned} \quad (29)$$

Now for the case  $m = n$

$$\begin{aligned} E [x_{i-m} x_{i-n}] &= E [x_{i-m}^2] = \frac{1}{\theta} \\ E [x_{i-m}^4] &= \frac{3}{\theta^2} \end{aligned}$$

The fourth moment is so because  $x_i$  is Gaussian.

Using the central limit theorem (holds as  $x_i$  are independent)

$$\begin{aligned} \sum_{i=0}^{K_c-1} \left( x_{i-m}^2 - \frac{1}{\theta} \right) &\sim N \left( 0, \frac{2K_c}{\theta^2} \right) \\ |C(m, m) - 1| &= \frac{\theta}{K_c} E \left| \sum_{i=0}^{K_c-1} \left( x_{i-m}^2 - \frac{1}{\theta} \right) \right| = \frac{2}{\sqrt{\pi K_c}} \end{aligned} \quad (30)$$

Taking (29) and (30) into (28) gives:

$$\begin{aligned}
& \sum_{m=1}^L \sum_{n=1}^L E |C(m, n) - \delta_{mn}| \\
&= LE |C(m, m) - 1| + (L^2 - L) E |C(m, n)|_{m \neq n} \\
&\leq \frac{2L}{\sqrt{\pi K_c}} + \frac{L^2 - L}{K_c}
\end{aligned}$$

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