Stat 200 HW7 Solutions

1 9.34

The MLE for θ is $\hat{\theta}_{MLE} = 0.0357$. Multiplying the resulting probabilities by the sample size $n = 3839$, we get the following table:

Type	Observed	Expected
Starchy green	1997	1953.76
Starchy white	906	925.49
Sugary green	904	925.49
Sugary white	32	34.26

Use Pearson's chi-square test, $X^2 = 2.015$. Under null hypothesis, X^2 is approximately chi-square distribution with 2 degrees of freedom. The p-value is about 0.365, and we will not reject null under $\alpha = .05$ test.

2 9.41

The likelihood function for this problem is:

$$\prod_{i=1}^{m} p_i^{X_i}(1-p_i)^{n_i-X_i}$$

The MLE for p_i is $\hat{p}_i = \frac{X_i}{n_i}$.
Under null hypothesis, all the $p_i's$ are equal, the likelihood function becomes:

$$p^{\sum X_i}(1-p)^{\sum(n_i-X_i)}$$

The MLE for p is $\hat{p} = \frac{\sum X_i}{\sum n_i}$. Thus, the likelihood ratio test statistic is:

$$\begin{aligned}\Lambda &= \frac{\hat{p}^{\sum X_i}(1-\hat{p})^{\sum(n_i-X_i)}}{\prod \hat{p}_i^{X_i}(1-\hat{p}_i)^{n_i-X_i}} \\ &= \frac{\hat{p}^{\sum n_i\hat{p}_i}(1-\hat{p})^{\sum n_i(1-\hat{p}_i)}}{\prod \hat{p}_i^{n_i\hat{p}_i}(1-\hat{p}_i)^{n_i(1-\hat{p}_i)}}\end{aligned}$$

By Theorem A in section 9.4,

$$-2\log\Lambda \sim \chi^2_{m-1}$$

under H_0.

3 9.43

(a) The Pearson chi-square statistic $X^2 = 11.99$. Under null, $X^2 \sim \chi^2_1$, and the p-value is 0.0005. Thus it is a significant discrepancy from the null.

Number of Heads	Observed	Expected
0	100	112.2
1	524	560.9
2	1080	1121.9
3	1126	1121.9
4	655	560.9
5	105	112.2

(b) Under null hypothesis, the probability of heads for all 5 coins are 1/2. Thus, if we tosses 5 coins at the same time, the number of heads has binomial distribution $\text{bin}(5, 0.5)$. Multiply the probabilities by the number of tossing, and get the expected observations in the following table.

The Pearson chi-square statistic is $X^2 = 21.57$. Under null, X^2 has a chi-square distribution with 5 df. The p-value is 0.0006, and the data are not consistent with the model.

(c) The MLE for p is:

$$\hat{p} = \frac{9207}{17950} = 0.513$$

Using same strategy as in previous part, we get $X^2 = 8.74$ with 4 df under null. The p-value is 0.068. The model is again doubtful, we will reject the model for level 0.1 test but not reject for level 0.05 test.

4 11.16

Let $X_1, \ldots, X_n$ be the treatment measurements, where the X_i are i.i.d. with mean μ_X and variance $\sigma^2 = 100$. Similarly let $Y_1, \ldots, Y_n$ be the control measurements, where the Y_i are i.i.d. with mean μ_Y and variance $\sigma^2 = 100$. We test the null hypothesis $H_0 : \mu_X - \mu_Y = 0$ vs. the one-sided alternative hypothesis $H_A : \mu_X - \mu_Y > 0$ using the test statistic

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{2\sigma^2/n}}$$

which, asymptotically, has a standard normal distribution under the null hypothesis. Then we reject H_0 in favor of H_A at level α when

$$Z > z_{1-\alpha}$$

where $z_{1-\alpha}$ is the $1 - \alpha$ quantile of a standard normal distribution. For $\alpha = 0.1$, we have $z_{1-\alpha} = 1.2816$. Now letting $\Delta = \mu_X - \mu_Y$ we have

$$\mathrm{E}(Z) = \frac{\Delta}{\sqrt{2\sigma^2/n}}$$

and so

$$Z - \frac{\Delta}{\sqrt{2\sigma^2/n}} \rightarrow_d \mathcal{N}(0, 1)$$

so we can compute the power under an alternative Δ as:

$$\begin{aligned} P_\Delta(Z > z_{1-\alpha}) &= P_\Delta\left(Z - \frac{\Delta}{\sqrt{2\sigma^2/n}} > z_{1-\alpha} - \frac{\Delta}{\sqrt{2\sigma^2/n}}\right) \\ &\approx 1 - \Phi\left(z_{1-\alpha} - \frac{\Delta}{\sqrt{2\sigma^2/n}}\right) \end{aligned}$$

Now substituting setting this expression to be equal to 0.5, and substituting in the values for $z_{1-\alpha}$ and Δ, we can solve for n:

$$\begin{aligned} & 0.5 = 1 - \Phi\left(1.2816 - \frac{2}{\sqrt{200/n}}\right) \\ \Rightarrow \quad & \Phi\left(1.2816 - \frac{2}{\sqrt{200/n}}\right) = 0.5 \\ \Rightarrow \quad & \left(1.2816 - \frac{2}{\sqrt{200/n}}\right) = \Phi^{-1}(0.5) \\ \Rightarrow \quad & \left(1.2816 - \frac{2}{\sqrt{200/n}}\right) = 0 \\ \Rightarrow \quad & \sqrt{n} = \frac{1.2816\sqrt{200}}{2} \\ \Rightarrow \quad & n = 82.1121 \end{aligned}$$

Thus n must be approximately 82 to achieve power of 0.5 at $\Delta = 2$.

5 11.35

Let $X_1, \ldots, X_{23}$ denote the observations from the control group, and let $Y_1, \ldots, Y_{22}$ denote the observations from the treatment group. A t-test comparing the control group to the ozone group with the hypothesis that the two groups have the same means may be performed as follows:

$$\begin{aligned} \bar{X} &= 22.426 & S_X &= 10.777 \\ \bar{Y} &= 11.009 & S_Y &= 19.017 \end{aligned}$$

so we find that

$$\begin{aligned} s_p^2 &= \frac{22S_X^2 + 21S_Y^2}{23 + 22 - 2} \\ &= 236.040 \end{aligned}$$

$$s_p = 15.364$$

Then the t statistic is given by

$$\begin{aligned} t &= \frac{\bar{X} - \bar{Y}}{s_p\sqrt{\frac{1}{n_X} + \frac{1}{n_Y}}} \\ &= 2.4919 \end{aligned}$$

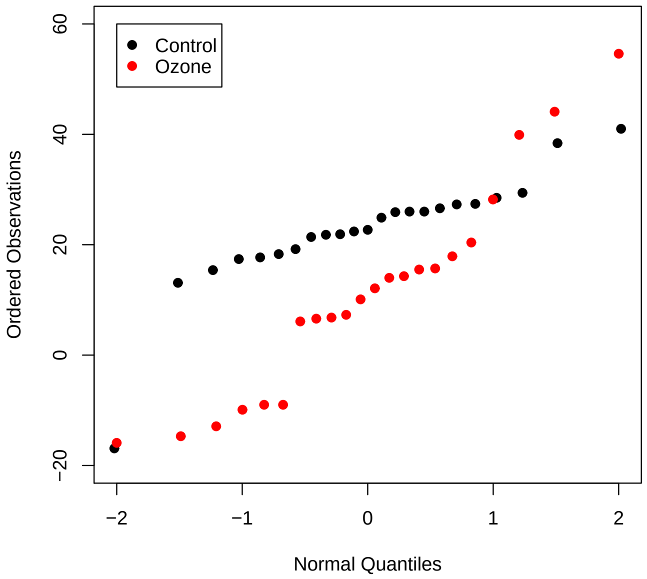

Figure 1: Normal quantile-quantile plots for the control and ozone groups.

Comparing this statistic to a t distribution with $22 + 23 - 2 = 43$ degrees of freedom, we find that for $T \sim t_{43}$

$$\begin{aligned} P(|T| > 2.4919) &= P(T > 2.4919) + P(T < -2.4919) \\ &= 2P(T < -2.4919) \\ &= 2(0.0083) \\ &= 0.0166 \end{aligned}$$

Thus, based on this p-value, we would reject the null hypothesis that the control group has the same mean as the ozone group for any level $\alpha > 0.0166$. However, note that in Figures $1 - 2$, we see some deviation from normality and some indication that the standard deviations might not be equal in the two groups. Therefore, we also consider the Mann-Whitney test.

Table 1 lists the observations and corresponding ranks in the two groups, and the total rank sum in each group. Since the group sizes exceed those listed in the table in the book, we must compute approximate critical values according to the given formula:

$$c_\alpha = \frac{n_1}{2}(n_1 + n_2 + 1) - z_{1-\alpha/2}\left[\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}\right]^{1/2}$$

This formula results in the critical values listed in Table 1 for the smaller rank sum in a two-sided test with $n_1 = 22, n_2 = 23$.

Then the null hypothesis that the two groups come from the same distribution is rejected at level α if the smaller of the two rank sums is less than the listed critical value c_α. Since for these data, the smaller of the two rank sums is 374, we would reject the null hypothesis for a two-sided test of level $\alpha = 0.005$, but not for a test of level $\alpha = 0.001$.

Note that the p-value resulting from a two-sided t-test is a bit higher than that of the Mann-Whitney test, but both are fairly low and therefore would generally lead to a rejection of the null hypothesis that the control and ozone groups come from the same distribution. Since the mean

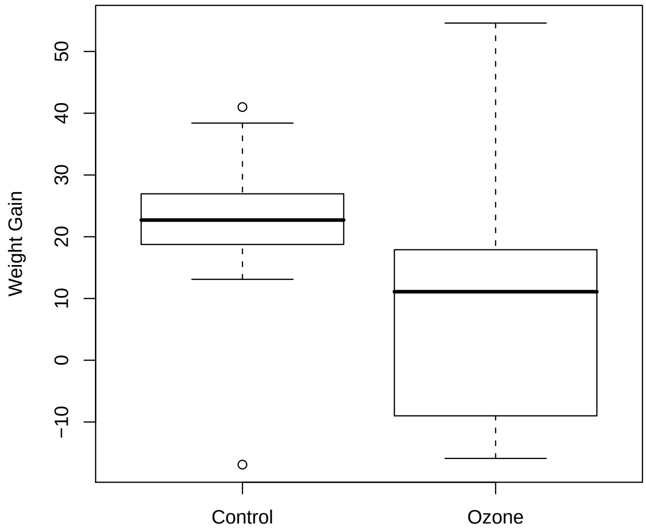

Figure 2: Boxplots of the weight gains for the control and ozone groups.

Control		Ozone	
Weight gain	Rank	Weight Gain	Rank
41.0	43	10.1	12
25.9	32	7.3	11
13.1	14	-9.9	5
-16.9	1	17.9	22
15.4	17	6.6	9
22.4	29	39.9	42
29.4	40	-14.7	3
26.0	33.5	-9.0	6.5
38.4	41	6.1	8
21.9	28	14.3	16
27.3	36	6.8	10
17.4	20	-12.9	4
27.4	37	12.1	13
17.7	21	-15.9	2
21.4	26	44.1	44
26.6	35	20.4	25
24.9	31	15.5	18
18.3	23	28.2	38
28.5	39	14.0	15
21.8	27	15.7	19
19.2	24	54.6	45
26.0	33.5	-9.0	6.5
22.7	30		
Rank Sum:	661	Rank Sum:	374

Table 1: Ranks for ozone and control observations

α	c_α
0.100	433.5579
0.050	419.6800
0.010	392.5562
0.005	382.3736
0.001	361.0798

Table 2: Approximate critical values for Mann-Whitney test with $n_1 = 22, n_2 = 23$

of the control group is higher, we would conclude that the rats who were exposed to ozone have significantly less weight gain than unexposed rats.

6 13.4

We conduct a chi–square test of homogeneity to determine if males mentioned the "being a college graduate" attribute significantly more than females did. First, compute constants

$$\begin{aligned} n_{1.} &= 141 \\ n_{2.} &= 643 \\ n_{.1} &= 369 \\ n_{.2} &= 415 \\ n_{..} &= 784 \end{aligned}$$

and expected values

$$\begin{aligned} E_{11} &= 66.36 \\ E_{12} &= 74.64 \\ E_{21} &= 302.64 \\ E_{22} &= 340.36 \end{aligned}$$

so that

$$\begin{aligned} X^2 &= \frac{(86-66.36)^2}{66.36} + \frac{(55-74.64)^2}{74.64} + \frac{(283-302.64)^2}{302.64} + \frac{(360-340.36)^2}{340.36} \\ &= 13.3884 \end{aligned}$$

with $(2-1)(2-1) = 1$ degree of freedom. The p–value is less than 0.001 and so we conclude that this difference between male responses and female responses is significant.

7 13.16

Conduct a chi–square test of independence to see if there is a relationship between personality type and attitude towards small cars. Compute:

$$\begin{aligned} n_{1.} &= 186 \\ n_{2.} &= 27 \\ n_{3.} &= 86 \\ n_{.1} &= 99 \\ n_{.2} &= 100 \\ n_{.3} &= 100 \\ n_{..} &= 299 \end{aligned}$$

and

$$\begin{aligned} E_{11} &= 61.59 \\ E_{12} &= 62.21 \\ E_{13} &= 62.21 \\ E_{21} &= 8.94 \\ E_{22} &= 9.03 \\ E_{23} &= 9.03 \\ E_{31} &= 28.47 \\ E_{32} &= 28.76 \\ E_{33} &= 28.76 \end{aligned}$$

Then,

$$X^2 = \sum \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = 27.29$$

with $(3-1)(3-1) = 4$ degrees of freedom. The p–value is less than 0.001 so we conclude that there is a relationship between personality style and attitude towards small cars. Moreover, it appears that cautious people more often than others view small cars favorably whereas explorers more often than others view small cars unfavorably.