

# Statistics 200: Homework 6 Solutions

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9.4 (a) The likelihood ratio values are:

$X$	$x_1$	$x_2$	$x_3$	$x_4$
$\Lambda$	2	3/4	3	1/2

thus ordering the  $x_i$  from smallest likelihood ratio value to largest, we get:

$$x_4, x_2, x_1, x_3.$$

- (b) The  $\alpha = 0.2$  test rejects if  $X = x_4$  and accepts otherwise. The  $\alpha = 0.5$  test rejects if  $X \in \{x_2, x_4\}$ .
- (c) We check to see which outcomes result in

$$\frac{P(x|H_0)}{p(x|H_A)} > 1.$$

From the table above, we see that outcomes  $x_1$  and  $x_3$  favor  $H_0$ .

- (d) We find prior probabilities  $P(H_0)$  and  $P(H_A)$  such that the test that rejects when

$$\frac{P(x|H_0)}{p(x|H_A)} < \frac{P(H_A)}{P(H_0)}$$

also rejects under the same conditions as the tests do in part (b). When  $\alpha = 0.2$ , the test rejects if

$$1/2 < P(H_A)/P(H_0)$$

thus  $P(H_A) + P(H_0) = 1$  implies  $P(H_A) > 1/3$  and  $P(H_0) < 2/3$ . However, if  $P(H_A)$  is too high, the test will reject too often; therefore the test must also satisfy

$$3/4 > P(H_A)/P(H_0).$$

Hence, we get the intervals  $1/3 < P(H_A) < 3/7$  and  $4/7 < P(H_0) < 2/3$ .

When  $\alpha = 0.5$  we satisfy

$$\begin{aligned} 3/4 &< P(H_A)/P(H_0) \\ 2 &> P(H_A)/P(H_0). \end{aligned}$$

so that  $3/7 < P(H_A) < 2/3$  and  $P(1/3 < H_0) < 4/7$ .

9.6 (a) Let  $H_0 : p = 0.5$  and  $H_1 : p = 0.7$ . We find values of  $X$  such that

$$\frac{P(x|H_0)}{P(x|H_A)} > 1.$$

Then,

$$\frac{0.5(0.5)^{x-1}}{0.7(0.3)^{x-1}} > 1$$

which implies

$$x > \frac{\log(7/5)}{\log(5/3)} + 1 = 1.658.$$

Hence, outcomes  $X \geq 2$  favor  $H_0$  and the outcome  $X = 1$  favors  $H_1$ .

(b) Now we favor  $H_0$  when

$$\frac{0.5(0.5)^{x-1}}{0.7(0.3)^{x-1}} = \frac{P(x|H_0)}{P(x|H_A)} > \frac{1}{10},$$

so

$$x > 0 > \frac{\log(7/50)}{\log(5/3)} + 1$$

which suggests that any observation favors  $H_0$ .

(c) Under  $H_0$ ,

$$P(X \geq 8|H_0) = 1 - \sum_{i=1}^7 0.5(0.5)^{i-1} = 0.007813.$$

(d) The power is

$$P(X \geq 8|H_1) = 1 - \sum_{i=1}^7 0.7(0.3)^{i-1} = 0.000219.$$

This test performs very poorly.

9.18 The likelihood ratio is given by

$$\begin{aligned} LR &= \frac{\prod_{i=1}^n \frac{1}{2} \lambda_0 e^{-\lambda_0 |x_i|}}{\prod_{i=1}^n \frac{1}{2} \lambda_1 e^{-\lambda_1 |x_i|}} \\ &= \left( \frac{\lambda_0}{\lambda_1} \right)^n e^{(\lambda_1 - \lambda_0) \sum_{i=1}^n |x_i|}, \end{aligned}$$

and we reject when the ratio is small. Because  $\lambda_0$  and  $\lambda_1$  are fixed with  $\lambda_1 - \lambda_0 > 0$ , this test is equivalent to

$$\sum_{i=1}^n |x_i| < c$$

for some  $c$  such that  $P_{H_0}(\sum_{i=1}^n |X_i| < c) = \alpha$ . Consider that  $X \sim DExp(\lambda_0)$  implies  $|X| \sim Exp(\lambda_0)$  so that the sum  $\sum_{i=1}^n |X_i| \sim Exp(\lambda_0/n)$ . Thus,  $c = z_\alpha(\lambda_0/n)$ , the  $\alpha$  quantile of the exponential distribution with parameter  $\lambda_0/n$ . Noting that  $c$  does not depend on  $\lambda_1$  and that our method for deriving this test would be the same for any  $\lambda > \lambda_0$ , this test is uniformly most powerful.

- 9.20 The most powerful test is found using the Neyman–Pearson Lemma and is a likelihood ratio test with

$$\frac{1}{2x} < c \Rightarrow x > \frac{1}{2c}.$$

With significance level 0.10, we solve  $P_{f_0}(X > \frac{1}{2c}) = 0.1$  which implies  $\frac{1}{2c} = 0.9$ . The power of this test is

$$P_{H_1}(X > 0.9) = \int_{0.9}^1 2x^2 dx = 0.18.$$

- 9.28 (a) This is a two-tailed test thus the p-value is

$$p = P(|T| > 1.5) = 2P(T > 1.5) = 0.134.$$

- (b) This is a one-tailed test so the p-value is

$$p = P(T > 1.5) = 0.067.$$

- 9.30 (a) When we observe a test statistic  $T_{obs}$ , the p-value is

$$p(T_{obs}) = P(T > T_{obs}) = 1 - F(T_{obs}).$$

Thus, the p-value is a function of the outcome of  $T$ , hence

$$V = 1 - F(T).$$

- (b) By proposition C,  $F(T)$  is a uniform random variable on  $(0, 1)$ , then  $1 - F(T)$  is uniform on  $(0, 1)$  as well.  
(c) Under the null hypothesis, the p-value is uniform on  $(0, 1)$  and therefore has a 0.90 probability of being greater than 0.1.  
(d) Note

$$V < \alpha \iff 1 - F(T) < \alpha$$

i.e the test rejects when  $P(T > T_{obs}) < \alpha$ . Since this event has probability  $\alpha$ , the significance level of the test is  $\alpha$ .

9.32 (a) The likelihood ratio is given by

$$\frac{lik_A}{lik_B} = \frac{\frac{1}{2\pi(25)} e^{-\frac{(x-100)^2}{2(25)^2}}}{\frac{1}{2\pi(25)} e^{-\frac{(x-125)^2}{2(25)^2}}}$$

so that if  $X = 120$ , this ratio becomes  $e^{-3/10}$ .

(b) With the prior probabilities equal, we solve

$$\frac{P(A|X)}{P(B|X)} = \frac{P(X|A)}{P(X|B)}$$

so that  $P(A|X) = e^{-3/10}P(B|X)$  and with  $P(A|X) + P(B|X) = 1$ .  
Thus,

$$e^{-3/10}P(B|X) + P(B|X) = 1$$

so that

$$P(B|X) = \frac{1}{1 + e^{-3/10}}.$$

- (c) This is the probability  $P(X > 125|A)$ , hence the probability that  $X$  is at least one standard deviation above the mean, which is about 0.16.
- (d) The power is  $P(X > 125|B) = 0.5$ , since we reject if  $X$  takes on a value greater than the mean with a normal distribution.
- (e) The p-value is  $P(X > 120|A) = P\left(\frac{X-100}{25} > \frac{120-100}{25}\right) = P(Z > 4/5)$ , where  $Z$  is a standard normal random variable. This probability is about 0.2119.