# Stat 200 HW4 Solution

## 1 8.1

The average count in 1-second intervals is .8392, so we choose it to be the estimator of $\lambda$. Apply the same approach described in Section 8.2, we get the following table:

| n | Observed | Expected |
|---|---|---|
| 0 | 5267 | 5257.7 |
| 1 | 4436 | 4412.3 |
| 2 | 1800 | 1851.4 |
| 3 | 534 | 517.9 |
| 4 | 111 | 108.7 |
| 5+ | 21 | 21. 1 |
| | 12169 | 12169.1 |

Qualitatively, the expected counts and observed counts matched pretty well.

## 2 8.5

**(a)** Use the methods of moment, the 1st moment is:

$$\mu_1 = EX = \theta + 2(1-\theta) = 2 - \theta$$
$$\Rightarrow \theta = 2 - \mu_1$$

The 1st sample moment is:

$$\hat{\mu}_1 = \frac{x_1 + x_2 + x_3}{3} = \frac{1+2+2}{3} = \frac{5}{3}$$

So the estimate of $\theta$ is:

$$\hat{\theta} = 2 - \hat{\mu_1} = 2 - \frac{5}{3} = \frac{1}{3}$$

**(b)** The likelihood function is:

$$\text{lik}(\theta) = f(x_1, x_2, x_3|\theta) = \theta(1-\theta)^2$$

**(c)** The log likelihood function is:

$$l(\theta) = log(\theta) + 2log(1-\theta)$$

Setting the first order derivative to zero:

$$l'(\theta) = \frac{1}{\theta} - \frac{2}{1-\theta} = \frac{1-3\theta}{\theta(1-\theta)} = 0$$
$$\Rightarrow \hat{\theta}_{\text{mle}} = \frac{1}{3}$$

To check it indeed maximize the liklihood function, take the second order derivative:

$$l''(\theta) = -\frac{1}{\theta^2} - \frac{2}{(1-\theta)^2} < 0$$

**(d)** Now $f_{X|\Theta}(x|\theta) = \theta(1-\theta)^2$ and $f_\Theta(\theta) = 1_{[0,1]}$. Thus the posterior density is:

$$\begin{aligned}
f_{\Theta|X}(\theta|x) &= \frac{f_{X|\Theta}(x|\theta)f_\Theta(\theta)}{\int f_{X|\Theta}(x|\theta)f_\Theta(\theta)\mathrm{d}\theta} \\
&= \frac{\theta(1-\theta)^2}{\int_0^1 \theta(1-\theta)^2\mathrm{d}\theta} \\
&= \frac{\theta(1-\theta)^2}{\int_0^1 (\theta - 2\theta^2 + \theta^3)\mathrm{d}\theta} \\
&= \frac{\theta(1-\theta)^2}{\frac{1}{2}\theta^2|_0^1 - \frac{2}{3}\theta^3|_0^1 + \frac{1}{4}\theta^4|_0^1} \\
&= 12\theta(1-\theta)^2, \text{ for } \theta \in [0,1]
\end{aligned}$$

# 3 8.13

**(a)**

$$\begin{aligned}
E(\hat{\alpha}) &= 3E(\bar{X}) \\
&= \frac{3}{n}E(X_1 + ... + X_n) \\
&= \frac{3}{n}\frac{n\alpha}{3} \\
&= \alpha
\end{aligned}$$

**(b)**

$$\begin{aligned}
Var(\hat{\alpha}) &= Var(3\bar{X}) \\
&= \frac{9}{n^2}Var(X_1 + ... + X_n) \\
&= \frac{9}{n}Var(X_1) \\
&= \frac{9}{n}\left[\int_{-1}^{1} x^2\frac{1+\alpha x}{2}\mathrm{d}x - \mu^2\right] \\
&= \frac{9}{n}\left(\frac{1}{3} - \frac{\alpha^2}{9}\right) \\
&= \frac{3-\alpha^2}{n}
\end{aligned}$$

**(c)** By CLT,

$$\frac{\hat{\alpha} - \alpha}{\sqrt{\frac{3-\alpha^2}{n}}} \sim N(0,1)$$

If $n = 25$ and $\alpha = 0$, then:

$$P(|\hat{\alpha}| > .5) = P(\frac{|\hat{\alpha}|}{\sqrt{3/25}} > \frac{.5}{\sqrt{3/25}})$$

$$
\begin{aligned}
&= 2(1-\Phi(\frac{.5}{\sqrt{3/25}})) \\
&= .1489
\end{aligned}
$$

# 4 8.16

**(a)** Observe that the density function $f(x|\sigma)$ is even, we have $\mu_1 = 0$. The second moment is:

$$
\begin{aligned}
\mu_2 &= \int_{-\infty}^{\infty} x^2 f(x|\sigma)\mathrm{d}x \\
&= 2\int_0^{\infty} \frac{x^2}{2\sigma} e^{-x/\sigma}\mathrm{d}x \\
&= \int_0^{\infty} \frac{x^2}{\sigma} e^{-x/\sigma}\mathrm{d}x \\
&= 2\sigma^2
\end{aligned}
$$

The last step uses the fact that the mean and variance of $Exp(\sigma^{-1})$ are $\sigma$ and $\sigma^2$ respectively. Thus estimator of $\sigma$ using methods of moments is:

$$
\hat{\sigma} = \sqrt{\hat{\mu}_2/2} = \sqrt{\frac{x_1^2 + \ldots + x_n^2}{2n}}
$$

**(b)** The log liklihood function is:

$$
l(\sigma) = -\frac{\sum_{i=1}^n |x_i|}{\sigma} - n\log(2\sigma)
$$

Setting the first order derivative to zero and solving for the mle:

$$
\begin{aligned}
l'(\sigma) &= \frac{\sum_{i=1}^n |x_i|}{\sigma^2} - \frac{n}{\sigma} = 0 \\
\Rightarrow \hat{\sigma} &= \frac{\sum_{i=1}^n |x_i|}{n}
\end{aligned}
$$

**(c)** First, compute the information $I(\sigma)$:

$$
\begin{aligned}
I(\sigma) &= -E\left[\frac{\partial^2}{\partial\sigma^2}\log f(X|\sigma)\right] \\
&= E\left[\frac{2|X|}{\sigma^3} - \frac{1}{\sigma^2}\right] \\
&= \int_{-\infty}^{\infty} \frac{2|x|}{\sigma^3} f(x|\sigma)\mathrm{d}x - \frac{1}{\sigma^2} \\
&= 2\int_0^{\infty} \frac{2x}{2\sigma^4} exp\{-\frac{x}{\sigma}\}\mathrm{d}x - \frac{1}{\sigma^2} \\
&= \frac{2}{\sigma^2} - \frac{1}{\sigma^2} \qquad (1) \\
&= \frac{1}{\sigma^2}
\end{aligned}
$$

Here, (1) uses the property of exponential distributions.

Thus the asymptotic variance of mle is $\frac{1}{nI(\sigma)} = \frac{\sigma^2}{n}$.

## 5 8.48

Let $g(x) = -log(x)$, first use first order Taylor expansion to find the approximate variance:

$$\tilde{\lambda} - \lambda = g(Y/n) - g(p_0) \approx g'(p_0)(Y/n - p_0)$$

$$Var(\tilde{\lambda}) \approx Var(g'(p_0)(Y/n - p_0)) = (g'(p_0))^2 \frac{p_0(1-p_0)}{n} = \frac{1-p_0}{np_0}$$

Now we use second order Taylor expansion to find the bias:

$$\tilde{\lambda} - \lambda = g(Y/n) - g(p_0) \approx g'(p_0)(Y/n - p_0) + \frac{g''(p_0)}{2}(Y/n - p_0)^2$$

$$bias(\tilde{\lambda}) \approx E[\frac{g''(p_0)}{2}(Y/n - p_0)^2] = \frac{1-p_0}{2np_0}$$

Example A in section 8.5 shows that the mle for $\lambda$ is $\bar{X}$, and has variance $\frac{\lambda}{n}$. So the efficiency of $\tilde{\lambda}$ relative to $\hat{\lambda}$ is:

$$\text{eff}(\tilde{\theta}, \hat{\theta}) = \frac{Var(\tilde{\lambda})}{Var(\hat{\lambda})} = \frac{e^\lambda - 1}{\lambda}$$

We show numerical values for the efficiency for some $\lambda$'s below.

| $\lambda$ | 0.125 | 0.250 | 0.500 | 1.000 | 2.000 | 4.000 | 8.000 |
|---|---|---|---|---|---|---|---|
| eff | 1.065188 | 1.136102 | 1.297443 | 1.718282 | 3.194528 | 13.399538 | 372.494748 |

## 6 8.50

### (a)

$$\begin{aligned}
\mu_1 &= \int_0^\infty \frac{x^2}{\theta^2} e^{-x^2/(2\theta^2)} \mathrm{d}x \\
&= \frac{\sqrt{2\pi\theta^2}}{2\theta^2} \int_{-\infty}^\infty \frac{x^2}{\sqrt{2\pi\theta^2}} e^{-x^2/(2\theta^2)} \mathrm{d}x \quad (2) \\
&= \frac{\sqrt{2\pi\theta^2}}{2\theta^2} \theta^2 \\
&= \sqrt{\frac{\pi}{2}} \theta
\end{aligned}$$

In step (2), we use the fact that integrand is an even function and extend it to the real line. Thus the estimate of $\theta$ using method of moments is:

$$\hat{\theta}_{MM} = \sqrt{\frac{2}{\pi}} \hat{\mu}_1 = \sqrt{\frac{2}{\pi}} \bar{X}$$

**(b)** The log likelihood function is:

$$l(\theta) = -\sum \frac{x_i^2}{2\theta^2} + \sum \log x_i - 2n\log\theta$$

Setting the first derivative to zero and solving for mle:

$$l'(\theta) = \sum \frac{x_i^2}{\theta^3} - \frac{2n}{\theta} = 0$$

$$\Rightarrow \hat{\theta}_{mle} = \sqrt{\frac{\sum x_i^2}{2n}}$$

**(c)**

$$\begin{aligned} I(\theta) &= -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right] \\ &= E\left[\frac{3X^2}{\theta^4} - \frac{2}{\theta^2}\right] \\ &= \frac{3}{\theta^4}\int_0^\infty \frac{x^3}{\theta^2} e^{-x^2/(2\theta^2)} \mathrm{d}x - \frac{2}{\theta^2} \\ &= \frac{3}{\theta^2}\int_0^\infty 2ue^{-u}\mathrm{d}u - \frac{2}{\theta^2} \quad (\text{Let } u = \frac{x^2}{2\theta^2}) \\ &= \frac{6}{\theta^2} - \frac{2}{\theta^2} = \frac{4}{\theta^2} \end{aligned}$$

Thus the asymptotic variance of the mle is:

$$\frac{1}{nI(\theta)} = \frac{\theta^2}{4n}$$

## 7 8.55

**(a)** Let $X_1,\ X_2,\ X_3,\ X_4$ denote the counts. The log likelihood function is:

$$l(\theta) = -(X_1 + X_2 + X_3 + X_4)\log 4 + X_1\log(2+\theta) + (X_2 + X_3)\log(1-\theta) + X_4\log\theta$$

Setting the first order derivative to zero:

$$\begin{aligned} l'(\theta) &= \frac{X_1}{2+\theta} - \frac{X_2 + X_3}{1-\theta} + \frac{X_4}{\theta} \\ &= \frac{X_1\theta(1-\theta) - (X_2+X_3)\theta(2+\theta) + X_4(2+\theta)(1-\theta)}{\theta(1-\theta)(2+\theta)} \\ &= -\frac{(X_1+X_2+X_3+X_4)\theta^2 - (X_1 - 2X_2 - 2X_3 - X_4)\theta - 2X_4}{\theta(1-\theta)(2+\theta)} = 0 \end{aligned}$$

Therefore: $(X_1 + X_2 + X_3 + X_4)\theta^2 - (X_1 - 2X_2 - 2X_3 - X_4)\theta - 2X_4 = 0$. Solving this equation, the mle is the positive root of the quadratic equation.

And we can also compute the asymptotic variance $1/nI(\theta)$ as:

$$\begin{aligned} nI(\theta) &= E\left[\frac{X_1}{(2+\theta)^2} + \frac{X_2+X_3}{(1-\theta)^2} + \frac{X_4}{\theta^2}\right] \\ &= \frac{n}{4(\theta+2)} + \frac{n}{2(1-\theta)} + \frac{n}{4\theta} \end{aligned}$$

For the given data, we can compute: $\hat{\theta} = .0357$ and $s_{\hat{\theta}} = .0057$.

**(b)** Based on normality, the 95% CI for $\theta$ is simply $.0357 \pm .0112$.