# Statistics 200
# Winter 2009
# Homework 2 Solutions

4.6 $X$ is a continuous random variable with probability density function $f(x) = 2x, 0 \le x \le 1$.

(a) The expectation of $X$ is found by integrating $xf(x)$ over the interval $[0, 1]$ on which the density is non-zero.

$$\begin{aligned} E[X] &= \int_0^1 xf(x)\,dx \\ &= \int_0^1 2x^2\,dx \\ &= \frac{2}{3}x^3\Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

(b) For $Y = X^2$, we first calculate the density of $Y$ by finding the CDF then differentiating. Note that $0 \le X \le 1$ implies $0 \le Y \le 1$

$$\begin{aligned} P(Y \le y) &= P\left(X^2 \le y\right) \\ &= P\left(-\sqrt{y} \le X \le \sqrt{y}\right) \\ &= P\left(X \le \sqrt{y}\right) \qquad \text{(since } X \text{ must be positive)} \\ &= \int_0^{\sqrt{y}} 2x\,dx \\ &= x^2\Big|_0^{\sqrt{y}} \\ &= y, 0 \le y \le 1 \\ \Rightarrow f_Y(y) &= \frac{d}{dy}F_Y(Y) \\ &= 1, 0 \le y \le 1 \end{aligned}$$

Then to find the expectation of $Y$ we integrate $yf_Y(y)$ over the integral $[0, 1]$ on which the density of $Y$ is non-zero.

$$\begin{aligned}
E[Y] &= \int_0^1 y f(y)\, dy \\
&= \int_0^1 y\, dy \\
&= \frac{1}{2} y^2 \Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

(c) By Theorem A in Section 4.1.1, we have $E[g(X)] = \int g(x)f(x)\, dx$, so for $Y = g(X) = X^2$ we have:

$$\begin{aligned}
E[Y] &= \int_0^1 x^2 f(x)\, dx \\
&= \int_0^1 2x^3\, dx \\
&= \frac{2}{4} x^4 \Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

(d) The definition of the variance of a random variable $X$ is $\text{Var}(X) = E\left[(X - E[X])^2\right]$, so using Theorem A of Section 4.1.1 again with $g(X) = (X - E[X])^2$, we find that $\text{Var}(X)$ is:

$$\begin{aligned}
\text{Var}(X) &= E\left[(X - E[X])^2\right] \\
&= \int_0^1 \left(x - \frac{2}{3}\right)^2 2x\, dx \\
&= \int_0^1 \left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) 2x\, dx \\
&= \int_0^1 \left(2x^3 - \frac{8}{3}x^2 + \frac{8}{9}x\right) dx \\
&= \left(\frac{2}{4}x^4 - \frac{8}{9}x^3 + \frac{4}{9}x^2\right)\Big|_0^1 \\
&= \frac{1}{2} - \frac{4}{9} \\
&= \frac{1}{18}
\end{aligned}$$

Theorem B of Section 4.2 gives an alternate expression for the variance of $X$ as $\text{Var}(X) = E\left[X^2\right] - (E[X])^2$. We know $E\left[X^2\right] = E[Y] = \frac{1}{2}$ from parts (b) and (c), and we know $E[X] = \frac{2}{3}$ from part(a), so combining these results we have:

$$\begin{aligned}\text{Var}(X) &= E\left[X^2\right] - (E[X])^2 \\ &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{1}{18}\end{aligned}$$

4.28 Let $X$ be the number of aircraft hit by gunners, so we want to find $E[X]$. Define $X_i$ to be an indicator variable indicating the event that the $i^{th}$ aircraft is hit, so

$$X_i = \begin{cases} 1 & \text{if aircraft } i \text{ is hit} \\ 0 & \text{otherwise} \end{cases}$$

Then $X = \sum_{i=1}^n X_i$, and by linearity of expectation we have $E[X] = \sum_{i=1}^n E[X_i]$. Now we must find $E[X_i]$, which will be the same for all $i = 1, ..., n$ because the aircraft are exchangeable.

$$\begin{aligned}E[X_i] &= 1 \times P(X_i = 1) + 0 \times P(X_i = 0) \\ &= P(X_i = 1) \\ &= 1 - P(X_i = 0)\end{aligned}$$

Now define $Z_{ij}$ to be the indicator that gunner $j$ hits plane $i$, so

$$Z_{ij} = \begin{cases} 1 & \text{if gunner } j \text{ hits aircraft } i \\ 0 & \text{otherwise} \end{cases}$$

Clearly, the only way that plane $i$ is not hit is if none of gunners $j$ for $j = 1, \ldots, m$ hit plane $i$, so $\{X_i = 0\} \iff \{Z_{ij} = 0 \text{ for all } j = 1, \ldots, m\}$. We can easily calculate $P(Z_{ij} = 1)$ using the Law of Total Probability, letting $B_i$ denote the event that gunner $j$ selects aircraft $i$ as a target.

$$\begin{aligned}P\left(Z_{ij} = 1\right) &= P\left(Z_{ij} = 1 \mid B_i\right) P\left(B_i\right) + P\left(Z_{ij} = 1 \mid B_i^C\right) P\left(B_i^C\right) \\ &= p\frac{1}{n} + 0\left(1 - \frac{1}{n}\right) \\ &= \frac{p}{n} \\ \Rightarrow P\left(Z_{ij} = 0\right) &= 1 - P\left(Z_{ij} = 1\right) \\ &= 1 - \frac{p}{n}\end{aligned}$$

Then since the gunners choose and hit their targets independently, we have:

$$\begin{aligned}P\left(X_i = 0\right) &= P\left(Z_{ij} = 0, j = 1, \ldots m\right) \\ &= P\left(Z_{i1} = 0\right) P\left(Z_{i2} = 0\right) \ldots P\left(Z_{im} = 0\right) \\ &= \left(1 - \frac{p}{n}\right)^m\end{aligned}$$

So $E[X_i] = 1 - P(X_i = 0) = 1 - \left(1 - \frac{p}{n}\right)^m$, and therefore, substituting back into the original expression for $E[X]$,

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n} E[X_i] \\
&= n\left[1 - \left(1 - \frac{p}{n}\right)^m\right] \\
&= n - n\left(1 - \frac{p}{n}\right)^m
\end{aligned}$$

*[NOTE: This problem could also be solved by calculating $P(X_i = 1)$ directly, conditioning on the number of gunners who target plane $i$. This approach will result in a more complicated sum, but it is equally correct and ultimately reduces to the same answer.]*

4.42 For $X \sim Exp(\lambda)$, we have $E[X] = \frac{1}{\lambda}$, $\text{Var}(X) = \frac{1}{\lambda^2}$, and the CDF is given by $F_X(x) = 1 - e^{-\lambda x}$. We are told that the standard deviation of $X$ is $\sigma$, which means that

$$\sigma^2 = \text{Var}(X) = \frac{1}{\lambda^2} \qquad \Rightarrow \qquad \sigma = \frac{1}{\lambda}$$

Then we compute $P\left(|X - E[X]| > k\sigma\right)$, substituting $E[X] = \sigma$ as follows:

$$\begin{aligned}
P\left(|X - E[X]| > k\sigma\right) &= P\left(|X - \sigma| > k\sigma\right) \\
&= P\left(X - \sigma > k\sigma\right) + P\left(-(X - \sigma) > k\sigma\right) \\
&= P\left(X > (k+1)\sigma\right) + P\left(-X > (k-1)\sigma\right) \\
&= P\left(X > (k+1)\sigma\right) \qquad \text{(since } X > 0\text{, so } P(-X > (k-1)\sigma) = 0\text{)} \\
&= 1 - P\left(X \leq (k+1)\sigma\right) \\
&= 1 - \left(1 - e^{-\lambda(k+1)\sigma}\right) \\
&= e^{-\lambda(k+1)\frac{1}{\lambda}} \\
&= e^{-(k+1)}
\end{aligned}$$

Using Chebyshev's inequality, we have $P\left(|X - E[X]| > t\right) \leq \frac{\sigma^2}{t^2}$. Substituting in $E[X] = \sigma$ and letting $t = k\sigma$, we find:

$$\begin{aligned}
P\left(|X - \sigma| > k\sigma\right) &\leq \frac{\sigma^2}{(k\sigma)^2} \\
&= \frac{1}{k^2}
\end{aligned}$$

The values of the exact probability and the Chebyshev bound are given below.

| $k$ | Exact Probability | Chebyshev Bound |
|---|---|---|
| 2 | 0.0498 | 0.2500 |
| 3 | 0.0183 | 0.1111 |
| 4 | 0.0067 | 0.0625 |

4.52 (a) The first security is the better choice because it has a higher expected return and a lower risk.

(b) The expected return is given by $E[R(\pi)]$ where $\pi = 0.5$:

$$\begin{aligned} E[R(\pi)] &= \pi\mu_1 \ + \ (1-\pi)\mu_2 \\ E[R(0.5)] &= 0.5(1) \ + \ 0.5(0.8) \\ &= 0.9 \end{aligned}$$

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.5$:

$$\begin{aligned} \text{Var}(R(\pi)) &= \pi^2\sigma_1^2 \ + \ 2\pi(1-\pi)\rho\sigma_1\sigma^2 \ + \ (1-\pi)^2\sigma_2^2 \\ \text{Var}(R(0.5)) &= 0.5^2\left(0.1^2\right) \ + \ 2\,(0.5)\,(0.5)\,(-0.8)\,(0.1)\,(0.12) \ + \ 0.5^2\left(0.12^2\right) \\ &= 0.0013 \\ \Rightarrow \sqrt{\text{Var}(R(0.5))} &= 0.0361 \end{aligned}$$

(c) The expected return is given by $E[R(\pi)]$ where $\pi = 0.8$:

$$\begin{aligned} E[R(\pi)] &= \pi\mu_1 \ + \ (1-\pi)\mu_2 \\ E[R(0.8)] &= 0.8(1) \ + \ 0.2(0.8) \\ &= 0.96 \end{aligned}$$

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.8$:

$$\begin{aligned} \text{Var}(R(\pi)) &= \pi^2\sigma_1^2 \ + \ 2\pi(1-\pi)\rho\sigma_1\sigma^2 \ + \ (1-\pi)^2\sigma_2^2 \\ \text{Var}(R(0.8)) &= 0.8^2\left(0.1^2\right) \ + \ 2\,(0.8)\,(0.2)\,(-0.8)\,(0.1)\,(0.12) \ + \ 0.2^2\left(0.12^2\right) \\ &= 0.0039 \\ \Rightarrow \sqrt{\text{Var}(R(0.5))} &= 0.0625 \end{aligned}$$

(d) The red line of Figure 1 displays the plot of $(\mu(\pi), \sigma(\pi))$ as $\pi$ varies from 0 to 1.

(e) The expected return is unchanged for different values of $\rho$, and is given by $E[R(\pi)]$ where $\pi = 0.5$:

$$E[R(0.5)] = 0.9$$

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.5$. When $\rho = 0.1$, we find:

$$\begin{aligned} \text{Var}(R(\pi)) &= \pi^2\sigma_1^2 \ + \ 2\pi(1-\pi)\rho\sigma_1\sigma^2 \ + \ (1-\pi)^2\sigma_2^2 \\ \text{Var}(R(0.5)) &= 0.5^2\left(0.1^2\right) \ + \ 2\,(0.5)\,(0.5)\,(0.1)\,(0.1)\,(0.12) \ + \ 0.5^2\left(0.12^2\right) \\ &= 0.0067 \\ \Rightarrow \sqrt{\text{Var}(R(0.5))} &= 0.0819 \end{aligned}$$

The expected return is unchanged for different values of $\rho$, and is given by $E[R(\pi)]$ where $\pi = 0.8$:

$$E[R(0.8)] = 0.96$$

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.8$. When $\rho = 0.1$ we find:

$$\begin{aligned} \text{Var}(R(\pi)) &= \pi^2\sigma_1^2 \ + \ 2\pi(1-\pi)\rho\sigma_1\sigma^2 \ + \ (1-\pi)^2\sigma_2^2 \\ \text{Var}(R(0.8)) &= 0.8^2\left(0.1^2\right) \ + \ 2\,(0.8)\,(0.2)\,(0.1)\,(0.1)\,(0.12) \ + \ 0.2^2\left(0.12^2\right) \\ &= 0.0074 \\ \Rightarrow \sqrt{\text{Var}(R(0.8))} &= 0.0858 \end{aligned}$$

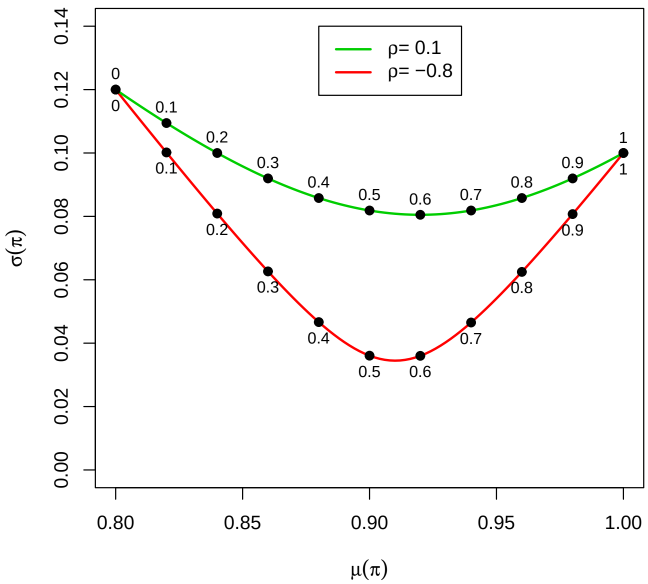


Figure 1: Plot of $(\mu(\pi), \sigma(\pi))$ as $\pi$ varies from 0 to 1. The red line displays the curve when $\rho = -0.8$, and the green line displays the curve when $\rho = 0.1$. The black points are $(\mu(\pi), \sigma(\pi))$ for the value of $\pi$ that is printed below or above the respective point.

The green line of Figure 1 displays the plot of $(\mu(\pi), \sigma(\pi))$ when $\rho = 0.1$ as $\pi$ varies from 0 to 1.

4.58 We are given $X_1 = f(x) + \varepsilon_1$ and $X_2 = f(x+h) + \varepsilon_2$ where $\varepsilon_1$ and $\varepsilon_2$ are independent with mean 0 and variance $\sigma^2$. The derivative $f'(x)$ is estimated by the random variable

$$Z = \frac{X_2 - X_1}{h}$$

(a) The expectation of $Z$ is found using the linearity of expectation property:

$$\begin{aligned} E[Z] &= E\left[\frac{X_2 - X_1}{h}\right] \\ &= \frac{1}{h}\left(E[X_2] - E[X_1]\right) \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

And the variance of $Z$ is found using the properties of a variance of a sum of independent random variables given on page 140 of the text:

$$\begin{aligned}
\text{Var}(Z) &= \text{Var}\left(\frac{X_2 - X_1}{h}\right) \\
&= \frac{1}{h^2}\left(\text{Var}(X_2) + \text{Var}(X_1)\right) \\
&= \frac{2\sigma^2}{h^2}
\end{aligned}$$

Clearly, $\text{Var}(Z) \to \infty$ as $h \to 0$, so choosing a very small value of $h$ will cause the variance to grow very large.

(b) The mean squared error ($MSE$) of an estimator $X$ for a quantity $x_0$ is given by

$$MSE(X) = E[(X - x_0)^2]$$

Therefore, the mean squared error of $Z$ as an estimator of $f'(x)$ is:

$$\begin{aligned}
MSE &= E[(X - f'(x))^2] \\
&= E[Z^2 - 2Zf'(x) + (f'(x))^2] \\
&= E[Z^2] - 2f'(x)E[Z] + (f'(x))^2
\end{aligned}$$

In order to proceed, we would need to know $f'(x)$, which we do not know. However, using a Taylor series expansion, we can derive an approximation. The Taylor series expansion of $f(x+h)$ about $x$ gives:

$$\begin{aligned}
&f(x+h) = f(x) + f'(x)(x + h - x) + \ldots \\
\Rightarrow &f(x+h) \approx f(x) + f'(x)(h) \\
\Rightarrow &f'(x) \approx \frac{f(x+h) - f(x)}{h} \\
\Rightarrow &f'(x) \approx E[Z]
\end{aligned}$$

Now plugging this approximation of $f'(x)$ back into our expression for the mean squared error of Z, we find:

$$\begin{aligned}
MSE &\approx E[Z^2] - 2E[Z]E[Z] + (E[Z])^2 \\
&= E[Z^2] - (E[Z])^2 \\
&= \text{Var}(Z) \\
&= \frac{2\sigma^2}{h^2} \qquad\qquad (MSE^*)
\end{aligned}$$

Thus $MSE^* \to 0$ as $h \to \infty$, so we cannot find a value of $h$ to minimize this approximation $MSE^*$ of the mean squared error.

(c) Now we are given three points, which we will express as

$$\begin{aligned}
X_1 &= f(x) + \varepsilon_1 \\
X_2 &= f(x+h) + \varepsilon_2 \\
X_3 &= f(x+h+k) + \varepsilon_3
\end{aligned}$$

again assuming that $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3$ are independent with mean 0 and common variance $\sigma^2$. Following the example of the first part of this problem, we define

$$\begin{aligned}
Z_1 &= \frac{X_2 - X_1}{h} \\
Z_2 &= \frac{X_3 - X_2}{k}
\end{aligned}$$

so $Z_1$ is an estimate of $g(x) = f'(x)$ and $Z_2$ is an estimate of $g(x+h) = f'(x+h)$. Again, we can analogously define

$$\begin{aligned} Y &= \frac{Z_2 - Z_1}{h} \\ &= \frac{1}{h}\left(\frac{X_3 - X_2}{k} - \frac{X_2 - X_1}{h}\right) \\ &= \frac{1}{hk}X_3 - \left(\frac{1}{hk} + \frac{1}{h^2}\right)X_2 + \frac{1}{h^2}X_1 \end{aligned}$$

which, by the same reasoning used in the set-up of the problem, is an estimate of $g'(x) = f''(x)$. Then we can compute the expectation and variance of $Y$:

$$\begin{aligned} E[Y] &= E\left[\frac{1}{hk}X_3 - \left(\frac{1}{hk} + \frac{1}{h^2}\right)X_2 + \frac{1}{h^2}X_1\right] \\ &= \frac{1}{hk}E[X_3] - \left(\frac{1}{hk} + \frac{1}{h^2}\right)E[X_2] + \frac{1}{h^2}E[X_1] \\ &= \frac{1}{hk}f(x+h+k) - \left(\frac{1}{hk} + \frac{1}{h^2}\right)f(x+h) + \frac{1}{h^2}f(x) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}\left(\frac{1}{hk}X_3 - \left(\frac{1}{hk} + \frac{1}{h^2}\right)X_2 + \frac{1}{h^2}X_1\right) \\ &= \frac{1}{(hk)^2}\mathrm{Var}(X_3) + \left(\frac{1}{hk} + \frac{1}{h^2}\right)^2 \mathrm{Var}(X_2) + \frac{1}{h^4}\mathrm{Var}(X_1) \\ &= \sigma^2\left(\frac{1}{h^2k^2} + \frac{1}{h^2k^2} + \frac{2}{h^3k} + \frac{1}{h^4} + \frac{1}{h^4}\right) \\ &= 2\sigma^2\left(\frac{1}{h^2k^2} + \frac{1}{h^3k} + \frac{1}{h^4}\right) \end{aligned}$$

4.102 We are given $\Theta = g(X,Y) = \tan^{-1}\left(\frac{Y}{X}\right)$, where $X$ has mean $x_0$ and variance $\sigma^2$, and $Y$ has mean $y_0$ and variance $\sigma^2$. Using the approximations given on page 165 of the text, we have:

$$E[g(X,Y)] \approx g(\mu_X, \mu_Y) + \frac{1}{2}\sigma_X^2 \frac{\partial g(\mu_X, \mu_Y)}{\partial x^2} + \frac{1}{2}\sigma_X^2 \frac{\partial g(\mu_X, \mu_Y)}{\partial y^2} + \sigma_{XY}\frac{\partial g(\mu_X, \mu_Y)}{\partial x \partial y}$$

$$\mathrm{Var}(g(X,Y)) \approx \sigma_X^2\left(\frac{\partial g(\mu_X, \mu_Y)}{\partial x}\right)^2 + \sigma_Y^2\left(\frac{\partial g(\mu_X, \mu_Y)}{\partial y}\right)^2 + 2\sigma_{XY}\left(\frac{\partial g(\mu_X, \mu_Y)}{\partial x}\right)\left(\frac{\partial g(\mu_X, \mu_Y)}{\partial y}\right)$$

In this problem, we are told that $X$ and $Y$ are independent, so the terms with $\sigma_{XY}$ disappear. We must now calculate all of the remaining partial derivatives:

$$\begin{aligned}\frac{\partial}{\partial x}g(x,y) &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)\\ &= \frac{-y}{x^2+y^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}g(x,y) &= (-y)(-1)\left(x^2+y^2\right)^{-2}(2x)\\ &= \frac{2xy}{\left(x^2+y^2\right)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}g(x,y) &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\\ &= \frac{1}{x+\frac{y}{x}}\\ &= \frac{x}{x^2+y^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial y^2}g(x,y) &= (x)(-1)\left(x^2+y^2\right)^{-2}(2y)\\ &= \frac{-2xy}{\left(x^2+y^2\right)^2}\end{aligned}$$

Plugging these results into the approximation expressions, we find:

$$\begin{aligned}E[\Theta] &\approx \tan^{-1}\left(\frac{y_0}{x_0}\right)+\frac{1}{2}\sigma^2\frac{2x_0y_0}{\left(x_0^2+y_0^2\right)^2}\;+\frac{1}{2}\sigma^2\frac{-2x_0y_0}{\left(x_0^2+y_0^2\right)^2}\\ &= \tan^{-1}\left(\frac{y_0}{x_0}\right)\\ \text{Var}(\Theta) &\approx \sigma^2\left(\frac{-y_0}{x_0^2+y_0^2}\right)^2+\sigma^2\left(\frac{x_0}{x_0^2+y_0^2}\right)^2\\ &= \sigma^2\left(\frac{y_0^2}{\left(x_0^2+y_0^2\right)^2}+\frac{x_0^2}{\left(x_0^2+y_0^2\right)^2}\right)\\ &= \frac{\sigma^2}{x_0^2+y_0^2}\end{aligned}$$

5.4 For $N \sim Poisson(\lambda)$, we have $E[N] = \text{Var}(N) = \lambda$. Here we are given that $E[N] = 100$, so $\lambda = 100$. To use the normal approximation to the Poisson, we must first standardize the Poisson random variable.

Let

$$Z = \frac{N - E[N]}{\sqrt{Var(N)}} = \frac{N - 100}{10}$$

So $Z$ has approximately the standard normal distribution. Using this fact, we compute:

$$\begin{aligned} P\left(100 - \Delta < N < 100 + \Delta\right) &= P\left(100 - \Delta - 100 < N - 100 < 100 + \Delta - 100\right) \\ &= P\left(\frac{-\Delta}{10} < \frac{N - 100}{10} < \frac{\Delta}{10}\right) \\ &= P\left(\frac{-\Delta}{10} < Z < \frac{\Delta}{10}\right) \\ &\approx \Phi\left(\frac{\Delta}{10}\right) - \Phi\left(\frac{-\Delta}{10}\right) \\ &= \Phi\left(\frac{\Delta}{10}\right) - \left[1 - \Phi\left(\frac{\Delta}{10}\right)\right] \end{aligned}$$

The last equality follows from the symmetry about 0 of the standard normal distribution, i.e., $\Phi(x) = 1 - \Phi(-x)$. Now we must solve for the value of $\Delta$ such that the above probability is 0.90:

$$\begin{aligned} &\Phi\left(\frac{\Delta}{10}\right) - \left[1 - \Phi\left(\frac{\Delta}{10}\right)\right] = 0.9 \\ &\Rightarrow 2\Phi\left(\frac{\Delta}{10}\right) - 1 = 0.9 \\ &\Rightarrow 2\Phi\left(\frac{\Delta}{10}\right) = 1.9 \\ &\Rightarrow \Phi\left(\frac{\Delta}{10}\right) = 0.95 \\ &\Rightarrow \frac{\Delta}{10} = 1.645 \qquad (*) \\ &\Rightarrow \Delta = 16.45 \end{aligned}$$

Here the step indicated by $(*)$ is performed using a table of standard normal quantiles.