STATS 200 Homework 1 Solution

Thanks to Jeremy Shen for providing parts of the solution.

1 Problem 1.60

Let $A_i = \{\text{The item is produced by } i^{th} \text{ shift}\}$ and $D = \{\text{The item is defective}\}$. Then:

$$P(D|A_1) = .01 \quad P(D|A_2) = .02 \quad P(D|A_3) = .05$$

The shifts all have the same productivity implies $P(A_i) = \frac{1}{3}$ for $i = 1, 2, 3$. Apply Law of total probability, we get:

$$P(D) = \sum_{i=1}^{3} P(D|A_i)P(A_i) = \frac{1}{3}(.01 + .02 + .05) = 2.67\%$$

The probability that a defective item was produced by the third shift is:

$$P(A_3|D) = \frac{P(D \cap A_3)}{P(D)} = \frac{P(D|A_3)P(A_3)}{P(D)} = .625$$

2 Problem 2.64

It is clearly that $E \geq 0$, for every $e > 0$,

$$\begin{aligned} F_E(E \leq e) &= P(E \leq e) \\ &= P(V^2/\sigma^2 \leq 2e/m\sigma^2) \\ &= P(-\sqrt{2e/m\sigma^2} \leq V/\sigma \leq \sqrt{2e/m\sigma^2}) \\ &= \Phi(\sqrt{2e/m\sigma^2}) - \Phi(-\sqrt{2e/m\sigma^2}) \end{aligned}$$

Differentiate with respect to e and use the chain rule, we get the density:

$$\begin{aligned} f_E(e) &= \frac{1}{\sqrt{2em\sigma^2}}\phi(\sqrt{2e/m\sigma^2}) + \frac{1}{\sqrt{2em\sigma^2}}\phi(-\sqrt{2e/m\sigma^2}) \\ &= \frac{1}{\sqrt{em\sigma^2\pi}} e^{-e/m\sigma^2} \quad \text{for } e > 0 \end{aligned}$$

3 Problem 3.20

The density of X_1 and $X_2|X_1$ are:

$$f_{X_1}(x_1) = 1_{\{x_i \in [0,1]\}} \text{ and } f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1} 1_{\{0 \leq x_2 \leq x_1 \leq 1\}}$$

respectively. Thus the joint pdf of $(X_1,\ X_2)$ is given by:

$$f(x_1,x_2) = f_{X_2|X_1}(x_2|x_1)f_{X_1}(x_1) = \frac{1}{x_1}1_{\{0\le x_2\le x_1\le 1\}}$$

For every $x_1, x_2 \in [0,1]$, the joint cdf is:

$$\begin{aligned}
F(x_1,x_2) &= \int\int_{[0,x_1]\times[0,x_2]} f(u,v)\mathrm{d}u\mathrm{d}v \\
&= \int_0^{x_1}\left(\int_0^{min\{x_2,u\}}\frac{1}{u}\mathrm{d}v\right)\mathrm{d}u \\
&= \int_0^{x_1}\frac{min\{x_2,u\}}{u}\mathrm{d}u \\
&= \begin{cases} \int_0^{x_1} 1\mathrm{d}u & , \quad \text{if } x_1\le x_2 \\ \int_0^{x_2} 1\mathrm{d}u + \int_{x_2}^{x_1}\frac{x_2}{u}\mathrm{d}u & , \quad \text{if } x_1 > x_2 \end{cases} \\
&= \begin{cases} x_1 & , \quad x_1\le x_2 \\ x_2 + x_2 log(x_1/x_2) & , \quad x_1 > x_2 \end{cases}
\end{aligned}$$

To get the marginal cdf, we use the joint cdf and set $x_2 = 1$ and $x_1 = 1$ respectively:

$$F_{X_1}(x_1) = F(x_1,1) = x_1$$

$$F_{X_2}(x_2) = F(1,x_2) = x_2 - x_2 log(x_2)$$

4 Problem 3.32

The posterior density $f_{\Theta|X}(\theta|x) = C\theta^x(1-\theta)^{n-x}$, where $C = \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)}$ is a constant when x is given. Differenciating with respect to θ and setting it to be 0, we have:

$$\begin{aligned}
f'_{\Theta|X}(\theta|x) &= xC\theta^{x-1}(1-\theta)^{n-x} - (n-x)C\theta^x(1-\theta)^{n-x-1} \\
&= C\theta^x(1-\theta)^{n-x}\left(\frac{x}{\theta} - \frac{n-x}{1-\theta}\right) \\
&= C\theta^{x-1}(1-\theta)^{n-x-1}(x-n\theta) = 0
\end{aligned}$$

If $x \ne 0, n$, then $C\theta^{x-1}(1-\theta)^{n-x-1} \ne 0$, thus $\hat{\theta} = \frac{x}{n}$ solves the equation. To see $f_{\Theta|X}(\theta|x)$ attains maximum when $\theta = \hat{\theta}$, we can check that:

$$f'_{\Theta|X}(\theta|x) = \begin{cases} >0 & , \quad \theta < \hat{\theta} \\ <0 & , \quad \theta > \hat{\theta} \end{cases}$$

If $x = 0$ or $x = n$, it is easy to see that $\hat{\theta}$ still maximize $f_{\Theta|X}(\theta|x)$ because:

$$f_{\Theta|X}(\theta|x) \le C = f_{\Theta|X}(\hat{\theta}|x)$$

The reslut makes sense since it is the proportion of heads in n trials.

5 Problem 3.33

(a). The following approach is fairly general in terms of finding the posterior density. We use the fact that $Posterior \propto Likelihood \times Prior$. Here the likelihood comes from geometric distribution:

$$f_{N|\Theta}(n|\theta) = \theta(1-\theta)^{n-1}$$

And of course the uniform prior is 1 on the interval $[0,1]$, and 0 elsewhere. So the posterior density:

$$f_{\Theta|N}(\theta|n) \propto \theta(1-\theta)^{n-1} \times 1_{\theta\in[0,1]}$$

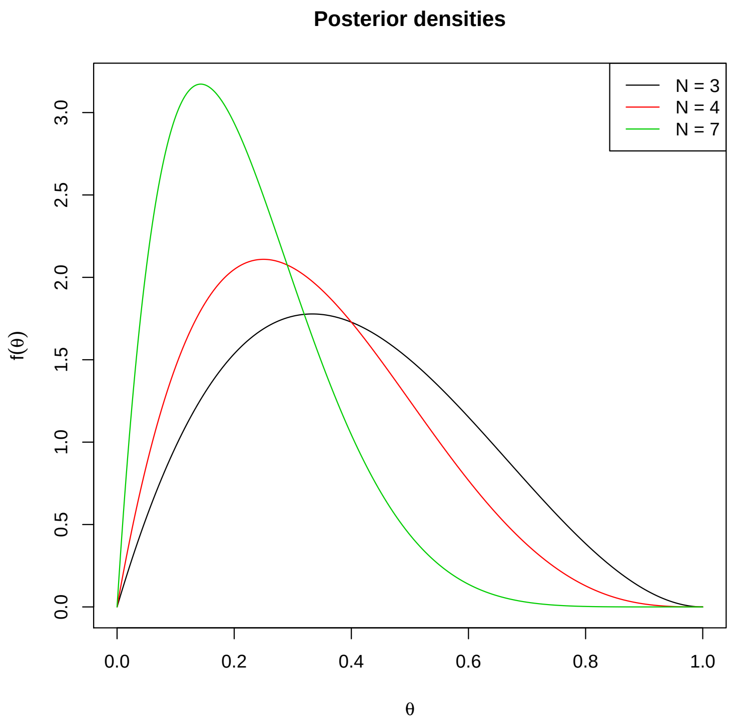
Where $1_{\theta\in[0,1]}$ is the indicator function. But recall that the posterior density is in θ and it must integrate to 1. To find the integrating constant, we can integrate by part:

$$\begin{aligned}
\int_0^1 \theta(1-\theta)^{n-1}d\theta &= -\frac{\theta}{n}(1-\theta)^n\Big|_0^1 + \int_0^1 \frac{(1-\theta)^n}{n}d\theta \\
&= -\frac{1}{n(n+1)}(1-\theta)^{(n-1)}\Big|_0^1 \\
&= \frac{1}{n(n+1)}
\end{aligned}$$

Since the posterior density should integrate to 1, we found the integrating constant to be n(n+1). We conclude that the posterior density is

$$f_{\Theta|N}(\theta|n) = n(n+1)\theta(1-\theta)^{n-1}, \text{ for } \theta \text{ in } [0,1]$$

(b). The answer will obvious vary, here we present the posterior densities for N = 3,4 and 7.



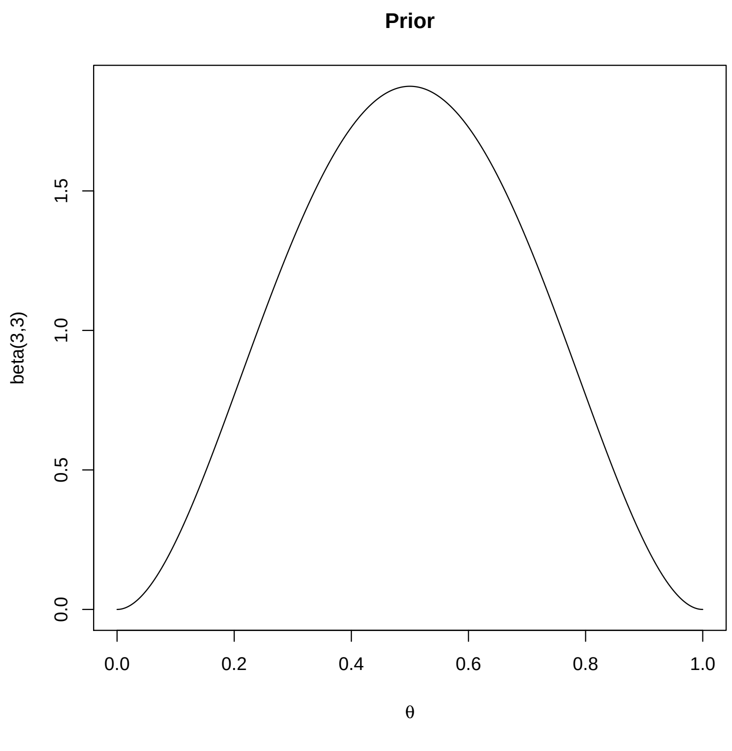
6 Problem 3.34

Recall the Beta(a,b) density:

$$g_U(u) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1}, \text{ for u} \in [0,1]$$

Consider when our prior belief about Θ is a Beta(3,3) distribution, with density:

$$f_\Theta(\theta) = \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} \theta^2(1-\theta)^2, \text{ for } \theta \in [0,1]$$



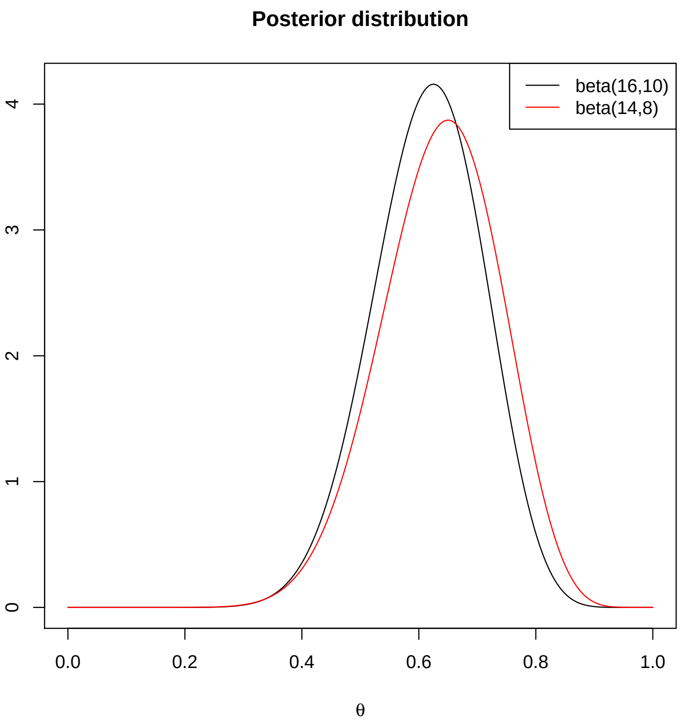
Using the likelihood $f_{X|\Theta}(x|\theta)$ from Example E of Section 3.5.2, we can compute the posterior density in the same way as in problem 3.33:

$$Posterior \propto \theta^x(1-\theta)^{n-x}\theta^2(1-\theta)^2 = \theta^{x+2}(1-\theta)^{n-x+2}$$

And conditional on x, we can identify this as the 'kernel' of a Beta(x+3, n-x+3) density, or more explicitly:

$$Posterior = f_{\Theta|X}(\theta|x) = \frac{\Gamma(n+6)}{\Gamma(x+3)\Gamma(n-x+3)} \theta^{x+2}(1-\theta)^{n-x+2}, \text{ for } \theta \in [0,1]$$

As in the example (p.95), take n=20, x = 13. Then we get a Beta(16, 10) density. Here is a plot comparing the textbook Beta(14,8) density with ours:



7 Problem 3.48

The density function of T_1 and T_2 are

$$f_{T_1}(t_1) = \lambda_1 e^{-\lambda_1 t_1} 1_{\{t_1 \geq 0\}} \quad \text{and} \quad f_{T_2}(t_2) = \lambda_2 e^{-\lambda_2 t_2} 1_{\{t_2 \geq 0\}}$$

respectively.

Let $T = T_1 + T_2$. Since T_1 and T_2 are independent, the density function of T is the convolution of the functions f_{T_1} and f_{T_2} (on p97), which is:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{T_1}(x) f_{T_2}(t-x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \lambda_1 e^{-\lambda_1 x} 1_{\{x \geq 0\}} \lambda_2 e^{-\lambda_2 (t-x)} 1_{\{t-x \geq 0\}} \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \lambda_1 \lambda_2 e^{-\lambda_2 t} e^{-(\lambda_1 - \lambda_2)x} 1_{\{0 \leq x \leq t\}} \mathrm{d}x \\ &= \lambda_1 \lambda_2 e^{-\lambda_2 t} 1_{\{t \geq 0\}} \int_0^t e^{-(\lambda_1 - \lambda_2)x} \mathrm{d}x \\ &= \begin{cases} \lambda_1^2 t e^{-\lambda_1 t} 1_{\{t \geq 0\}} & , \quad \lambda_1 = \lambda_2 \\ \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} 1_{\{t \geq 0\}} \left[e^{-\lambda_2 t} - e^{-\lambda_1 t} \right] & , \quad \lambda_1 \neq \lambda_2 \end{cases} \end{aligned}$$