STAT 215 PROBLEM SET 2

(1) Three different Markov chains are defined by the following transition matrices:

(a) $$\begin{pmatrix} 1/3 & 1/4 & 0 & 1/2 \\ 1/3 & 1/4 & 0 & 0 \\ 0 & 1/4 & 1 & 0 \\ 1/3 & 1/4 & 0 & 1/2 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1/2 & 1/3 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1 & 0 \\ 1/2 & 0 & 1/3 & 0 & 0 \end{pmatrix}$$

(c) $$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}$$

Draw a directed graph for each chain. Is the Markov chain irreducible? Identify the transient states, and the irreducible closed recurrent sets.

(2) Consider a Markov chain on the nonnegative integers such that, starting from x, the chain goes to state $x+1$ with probability p, $0 < p < 1$, and goes to state 0 ith probability $1-p$.

(a) Show that this chain is irreducible.

(b) Find $P_0(T_0 = n)$, $n \geq 1$.

(c) Show that the chain is recurrent.

(3) Consider an irreducible birth and death chain on the nonnegative integers. Show that if $p_x \leq q_x$ for $x \geq 1$, the chain is recurrent.

(4) Consider a branching chain with $f(0) = f(3) = 1/2$. Find the probability ρ of extinction.

(5) In a simple birth and death process, the birth rates are $p_i = bi$ and the death rates are $q_i = di$ for $i = 1, \ldots, N-1$. The birth and death rates are 0 elsewhere. The parameters b and d are positive and satisfy $(b+d)N \leq 1$. Let $\mu_n = E[X_n]$.

(a) Show that μ_n satisfies the following first-order difference equation:

$$\mu_{n+1} = (1+b-d)\mu_n + (d-b)NP(X_n = N).$$

(b) Show that

$$\mu_n \leq (1+b-d)^n \mu_0.$$

If $b < d$, find $\lim_{n\to\infty} \mu_n$.