Lecture 7: Random Effect Models

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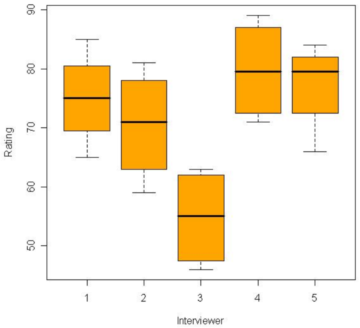
Announcements

- This week: Random effect models, variable transformations.
- Next Wednesday: In-class midterm.
- Midterm will cover material up to this Wednesday. You can bring one page of notes.
- Next Monday: PS2 due.

Example



Setting: Personnel management in a large enterprise.

Question: Does the interviewer have an effect on the rating of job candidates?

Data: 5 interviewers selected at random, each interviews 4 candidates selected at random.

Random Effects Model

Assuming that cell-sizes are the same, i.e. equal observations for each "subject" (brand of beer).

$$Y_{ij} \sim \mu_{\cdot} + \alpha_i + \varepsilon_{ij}, \quad 1 \leq i \leq r, 1 \leq j \leq n$$

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

$$\alpha_i \sim N(0, \sigma_\alpha^2)$$

Parameters:

- μ is the population mean;
- σ^2 is the measurement variance;
- σ_α^2 is the population variance of effect (i.e. variation in sodium content of beer).

One way ANOVA: r groups, n observations in each group.

- Fixed effect model:

Source	SS	df	$E(MS)$
Treatments	$SSTR = \sum_{i=1}^{r} n \left(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot}\right)^2$	$r-1$	$\sigma^2 + n\frac{\sum_{i=1}^{r} \alpha_i^2}{r-1}$
Error	$SSE = \sum_{i=1}^{r} \sum_{j=1}^{n} (Y_{ij} - \overline{Y}_{i\cdot})^2$	$(n-1)r$	σ^2

- Random effect model:

Source	SS	df	$E(MS)$
Treatments	$SSTR = \sum_{i=1}^{r} n \left(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot}\right)^2$	$r-1$	$\sigma^2 + n\sigma_\alpha^2$
Error	$SSE = \sum_{i=1}^{r} \sum_{j=1}^{n} (Y_{ij} - \overline{Y}_{i\cdot})^2$	$(n-1)r$	σ^2

One-way ANOVA (random)

Source	SS	df	$E(MS)$
Treatments	$SSTR = \sum_{i=1}^{r} n\left(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot}\right)^2$	$r-1$	$\sigma^2 + n\sigma_\alpha^2$
Error	$SSE = \sum_{i=1}^{r}\sum_{j=1}^{n}(Y_{ij} - \overline{Y}_{i\cdot})^2$	$(n-1)r$	σ^2

- Only change here is the expectation of *MSTR* which reflects randomness of α_i's.
- ANOVA table is still useful to setup tests: the same F statistics for fixed effect models will work here.
- Test for random effect: $H_0 : \sigma_\alpha^2 = 0$ based on

$$F = \frac{MSTR}{MSE} \sim F_{r-1,(n-1)r} \qquad \text{under } H_0.$$

Estimating $\sigma_\alpha^2/(\sigma_\alpha^2+\sigma^2)$

As before, the MSTR and MSE are independent.

$$\frac{MSTR/(n\sigma_\alpha^2+\sigma^2)}{MSE/\sigma^2} \sim F_{r-1,r(n-1)},$$

Thus,

$$P\left\{F_{r-1,r(n-1)}(\alpha/2) \leq \frac{MSTR/(n\sigma_\alpha^2+\sigma^2)}{MSE/\sigma^2} \leq F_{r-1,r(n-1)}(1-\alpha/2)\right\} = 1-\alpha.$$

Rearranging, we have:

$$L = \frac{1}{n}\left[\frac{MSTR}{MSE}\frac{1}{F(1-\alpha/2)} - 1\right]$$

$$U = \frac{1}{n}\left[\frac{MSTR}{MSE}\frac{1}{F(\alpha/2)} - 1\right]$$

are lower and upper confidence bounds for σ_α^2/σ^2. For $\sigma_\alpha^2/(\sigma_\alpha^2+\sigma^2)$, they are:

$$L^* = \frac{L}{1+L} \quad L^* = \frac{U}{1+U}.$$

Estimating σ_α^2

- From the ANOVA table

$$\sigma_\alpha^2 = \frac{E(SSTR/(r-1)) - E(SSE/((n-1)r))}{n}.$$

- Natural estimate:

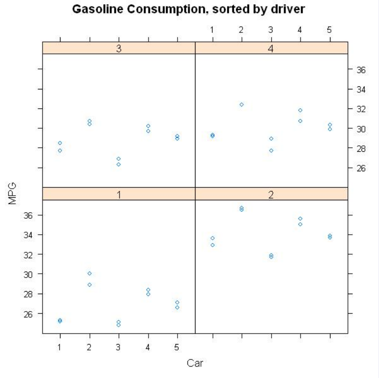
$$S_\alpha^2 = \frac{SSTR/(r-1) - SSE/((n-1)r)}{n}$$

- Problem: this estimate can be negative. If it is, set to 0.

Two-way ANOVA (random)

An automobile manufacturer wishes to study the differences between drivers and between cars on gasoline consumption.

Example: Gasoline Consumption



Are there effects of:

- different cars
- different drivers

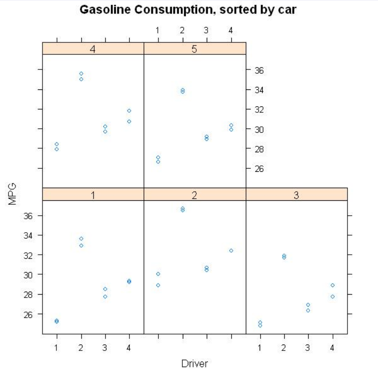
on gasoline consumption?

Data:

- 5 cars
- 4 drivers
- Each driver drove each car twice.

Two-way ANOVA (random)

Example: Gasoline Consumption



Are there effects of:

- different cars
- different drivers

on gasoline consumption?

Data:

- 5 cars
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- Each driver drove each car twice.

Two-way ANOVA (random)

Observations, for $1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq n$:

$$Y_{ijk} \sim \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},$$

$$\varepsilon_{ijk} \sim N(0, \sigma^2),$$

$$\alpha_i \sim N(0, \sigma^2_\alpha),$$

$$\beta_j \sim N(0, \sigma^2_\beta),$$

$$(\alpha\beta)_{ij} \sim N(0, \sigma^2_{\alpha\beta}).$$

Sums of squares

Identical to fixed effects models

$$SSA = nb \sum_{i=1}^{a} \left(\overline{Y}_{i..} - \overline{Y}_{...} \right)^2$$

$$SSB = na \sum_{j=1}^{b} \left(\overline{Y}_{.j.} - \overline{Y}_{...} \right)^2$$

$$SSAB = n \sum_{i=1}^{a} \sum_{j=1}^{b} \left(\overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...} \right)^2$$

ANOVA tables: Two-way (random)

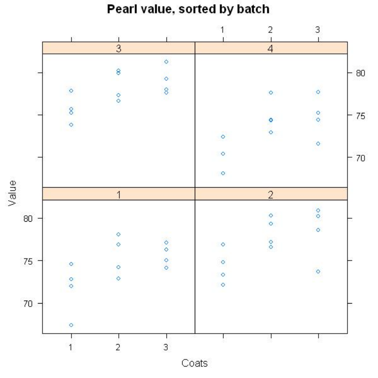
SS	df	$E(MS)$
SSA	$a-1$	$\sigma^2 + nb\sigma^2_\alpha + n\sigma^2_{\alpha\beta}$
SSB	$b-1$	$\sigma^2 + na\sigma^2_\beta + n\sigma^2_{\alpha\beta}$
$SSAB$	$(b-1)(a-1)$	$\sigma^2 + n\sigma^2_{\alpha\beta}$
SSE	$(n-1)ab$	σ^2

- To test $H_0 : \sigma^2_\alpha = 0$ use SSA and $SSAB$.
- To test $H_0 : \sigma^2_{\alpha\beta} = 0$ use $SSAB$ and SSE.

Two-way ANOVA (mixed)

Market research: How does number of coats of a special lacquer applied to a plastic bead affect its market value?

Example: Imitation Pearls



How does the number of lacquer coating on the pearl affect its market value?

Data:

- 4 batches of 12 beads each.
- 3 levels of coating (fixed in advance) applied to each batch.

Two-way ANOVA (mixed)

$$Y_{ijk} \sim \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, 1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq n$$

$$\varepsilon_{ijk} \sim N(0, \sigma^2),$$

$$\alpha_i \sim N(0, \sigma^2_\alpha),$$

$$\beta_j \quad \text{are constants.}$$

$$(\alpha\beta)_{ij} \sim N(0, (b-1)\sigma^2_{\alpha\beta}/b).$$

Constraints:

- $\sum_{j=1}^{b} \beta_j = 0$
- $\sum_{i=1}^{a} (\alpha\beta)_{ij} = 0, 1 \leq i \leq a.$

Details can safely be ignored here.

Two-way ANOVA

MS	df	Fixed	Random	Mixed
MSA	$a-1$	$\sigma^2 + nb\frac{\sum \alpha_i^2}{a-1}$	$\sigma^2 + nb\sigma_\alpha^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + na\sigma_\alpha^2$
MSB	$b-1$	$\sigma^2 + na\frac{\sum \beta_j^2}{b-1}$	$\sigma^2 + na\sigma_\beta^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + nb\frac{\sum_{j=1}^b \beta_i^2}{b-1} + n\sigma_{\alpha\beta}^2$
$MSAB$	$(b-1)(a-1)$	$\sigma^2 + n\frac{\sum\sum(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$	$\sigma^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + n\sigma_{\alpha\beta}^2$
MSE	$(n-1)ab$	σ^2	σ^2	σ^2

Mixed Effect Models:

- To test $H_0 : \sigma_\alpha^2 = 0$ use MSA and MSE.
- To test $H_0 : \beta_1 = \cdots = \beta_b = 0$ use MSB and $MSAB$.
- To test $H_0 : \sigma_{\alpha\beta}^2$ use $MSAB$ and MSE.

The `R` library for fitting random and mixed effect models is called `nlme`. You can use the function `lme` from this library. We are only going to skim the surface here.

For `lme`, you need to give what `R` calls a model formulae:

- `miles.lme = lme(MPG~ 1, data=miles,random=~ 1|Driver/Car)`
- `pearls.lme = lme(Value~ Coats, data=pearls,random=~ 1|Batch)`