Math136/Stat219 Fall 2008
Sample Midterm

Write your name and sign the Honor code in the blue books provided.

You have 90 minutes to solve all questions, each worth points as marked (maximum of 50). Complete reasoning is required for full credit. You may cite lecture notes and homework sets, as needed, stating precisely the result you use, why and how it applies. You may consult the following materials while taking the exam:

1. Stat219/Math136 Lecture notes, Fall 2008 version (the required text)
2. Kevin Ross's Lecture slides posted in Coursework
3. Homework problems and solutions posted in Coursework
4. Your own graded homework papers
5. Your own notes taken during lecture

Use of any other material is prohibited and constitutes a violation of the Honor Code. This includes but is not limited to: other texts (including optional and recommended texts), photocopying of texts or notes, materials from previous sections of Stat219/Math136, the internet, programming formulas in a calculator or computer, consultation with anyone during the exam (except for the Instructor).

1.(4x2) Let $\Omega = I\!R$ with event space $\mathcal{B}$ and the uniform probability measure U on $(0,1)$.

a) Find the distribution functions $F_X(x)$, $F_Y(y)$ and $F_V(v)$ that correspond to the random variables $X(\omega) = \omega$, $Y(\omega) = 0.5 I_A(\omega)$ and $V(\omega) = I_B(\omega)$, where $A = (0, 1/3)$ and $B = [1/4, 1/3)$.

ANS: Recall that $U((-\infty, x]) = x$ when $0 < x < 1$, with $U((-\infty, x]) = 1$ when $x \geq 1$ and $U((-\infty, x]) = 0$ when $x \leq 0$. Consequently, $F_X(x) = U((-\infty, x]) = x I_{[0,1)}(x) + I_{[1,\infty)}(x)$. We also know that $\mathbf{P}(Y = 0.5) = U(A) = 1/3$ and $\mathbf{P}(Y = 0) = U(A^c) = 2/3$. Hence, $F_Y(y) = (2/3) I_{[0,0.5)}(y) + I_{[0.5,\infty)}(y)$. Similarly, V is an indicator RV with $\mathbf{P}(V = 1) = U(B) = 1/3 - 1/4 = 1/12$, hence $F_V(v) = (11/12) I_{[0,1)}(v) + I_{[1,\infty)}(v)$.

b) Compute the characteristic function $\Phi_{\underline{Z}}(\theta_1, \theta_2)$ of the random vector $\underline{Z} = (X, Y)$.

ANS: By Definition 3.2.1, since $Y = 0.5 I_A(X)$ we have that

$$\Phi_{\underline{Z}}(\theta_1, \theta_2) = \mathbf{E}[\exp(i\theta_1 X + 0.5 i\theta_2 I_A(X))] = \mathbf{E}[I_{A^c}(X)e^{i\theta_1 X}] + e^{i\theta_2/2}\mathbf{E}[I_A(X)e^{i\theta_1 X}].$$

Recall that X has the density $f_X(x) = I_{(0,1)}(x)$, hence

$$\mathbf{E}[I_{A^c}(X)e^{i\theta_1 X}] = \int_{1/3}^{1} e^{i\theta_1 x} dx = (e^{i\theta_1} - e^{i\theta_1/3})/(i\theta_1)$$

(see text following Example 3.2.3 for similar calculations). Likewise, $\mathbf{E}[I_A(X)e^{i\theta_1 X}] = (e^{i\theta_1/3} - 1)/(i\theta_1)$, so that

$$\Phi_{\underline{Z}}(\theta_1, \theta_2) = (e^{i\theta_1} - e^{i\theta_1/3})/(i\theta_1) + e^{i\theta_2/2}(e^{i\theta_1/3} - 1)/(i\theta_1).$$

2.(4x3) Consider two independent tosses of a fair coin. Let $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^\Omega$, and $\mathbf{P}$ be the probability measure which assigns probability 1/4 to each $\omega \in \Omega$. Let X be the random variable which counts the number of heads in the two tosses, and let the random variable Y be 1 if the first toss results in a head and 0 otherwise.

a) Find $\sigma(X)$, $\sigma(Y)$, and $\sigma(X, Y)$.

ANS:

$$\begin{aligned}
\sigma(X) &= \sigma(\{\{HH\}, \{TT\}, \{HT, TH\}\}) \\
&= \{\emptyset, \Omega, \{HH\}, \{TT\}, \{HT, TH\}, \{HH, TT\}, \{HH\}^c, \{TT\}^c\}, \\
\sigma(Y) &= \{\emptyset, \Omega, \{HH, HT\}, \{TT, TH\}\}, \\
\sigma(X, Y) &= \sigma(\{\{HH\}, \{HT\}, \{TH\}, \{TT\}\}) = \mathcal{F}.
\end{aligned}$$

b) Find $\mathbf{E}(X|Y)$ (be sure to justify your answer rigorously).

ANS: It seems reasonable to guess that $\mathbf{E}(X|Y) = 1.5 I_{\{Y=1\}} + 0.5 I_{\{Y=0\}} (= Y + 0.5)$, or equivalently, $\mathbf{E}(X|Y) = 1.5 I_{\{HH,HT\}} + 0.5 I_{\{TT,TH\}}$. Clearly this is measurable with respect to $\sigma(Y)$. Thus we need to check the partial averaging condition for any $G \in \sigma(Y) = \{\emptyset, \Omega, \{HH, HT\}, \{TT, TH\}\}$. It holds trivially for $\emptyset$. Taking $G = \{HH, HT\}$ we have

$$\mathbf{E}(I_G(1.5 I_{\{HH,HT\}} + 0.5 I_{\{TT,TH\}})) = \mathbf{E}(1.5 I_{\{HH,HT\}}) = 1.5\mathbf{P}(\{HH, HT\}) = 0.75.$$

Also, $\mathbf{E}(I_G X) = 2\mathbf{P}(\{HH\}) + \mathbf{P}(\{HT\}) = 0.75$.

A similar calculation shows that for $G = \{TT, TH\}$,

$$\mathbf{E}(I_G(1.5 I_{\{HH,HT\}} + 0.5 I_{\{TT,TH\}})) = 0.25 = \mathbf{E}(I_G X)$$

Finally, the partial averaging condition holds for $G = \Omega$ by summing the results for $\{HH, HT\}$ and $\{TT, TH\}$.

Alternatively, define a RV Z which is 1 if the second toss results in a head and 0 otherwise. Then $X = Y + Z$. Also, since the two tosses are independent Y and Z are independent (you can also check this using the σ-fields $\sigma(Y)$ and $\sigma(Z) = \{\emptyset, \Omega, \{HH, TH\}, \{HT, TT\}$.) Then using the linearity of conditional expectation

$$\begin{aligned} \mathbf{E}(X|Y) = \mathbf{E}(Y + Z|Y) &= \mathbf{E}(Y|Y) + \mathbf{E}(Z|Y) \\ &= Y + \mathbf{E}(Z) = Y + 0.5, \end{aligned}$$

where the last line follows since Y is $\sigma(Y)$-measurable and Z is independent of $\sigma(Y)$.

c) Are $\sigma(X)$ and $\sigma(Y)$ independent? Why or why not?

ANS: For $\sigma(X)$ and $\sigma(Y)$ to be independent, we need $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1)\mathbf{P}(A_2)$ for all $A_1 \in \sigma(X)$ and $A_2 \in \sigma(Y)$. But, for example, letting $A_1 = \{HH\} \in \sigma(X)$ and $A_2 = \{HH, HT\} \in \sigma(Y)$, we have $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(\{HH\}) = 1/4$ and $\mathbf{P}(A_1)\mathbf{P}(A_2) = \mathbf{P}(\{HH\})\mathbf{P}(\{HH, HT\}) = (1/4)(1/2) = 1/8$. Thus $\sigma(X)$ and $\sigma(Y)$ are NOT independent.

3.(3x8) State which of the following statements is true and which is false. You get 1 point for each correct answer (-1 point for each wrong answer) + 2 points for its reasoning (that is, citing a specific result from lecture notes, deriving from known result or providing a counter example).

a) If a stochastic process $\{X_t, t \geq 0\}$ satisfies $\lim_{h\to 0} \mathbf{E}|X_{t+h} - X_t| = 0$ for all $t \geq 0$, then the stochastic process is continuous in probability.

TRUE: Using Markov's inequality, $\mathbf{P}(|X_{t+h} - X_t| > \epsilon) \leq (1/\epsilon)\mathbf{E}|X_{t+h} - X_t|$. Since for any $\epsilon > 0$ the LHS goes to 0 as $h \to 0$ (by assumption), so does the RHS, which is the definition of continuity in probability.

b) If X_n are nonnegative, integrable random variables for which $X_n \to 0$ a.s. then $\mathbf{E}(X_n) \to 0$ as $n \to \infty$.

FALSE: Consider $\Omega = [0, 1]$ with its Borel σ-field and the uniform probability measure. Let $X_n(\omega) = nI_{[0,1/n)}(\omega)$. Then $X_n(\omega) \to 0$ for all $\omega \in (0, 1]$, i.e. almost surely, but $\mathbf{E}(X_n) = 1$ for all n.

c) If X and Y are square-integrable random variables for which $\mathbf{E}(Y|X) = \mathbf{E}(Y)$, then X and Y are uncorrelated.

TRUE: Using the tower property and then taking out what is known, we have $\mathbf{E}(XY) = \mathbf{E}(\mathbf{E}(XY|X)) = \mathbf{E}(X\mathbf{E}(Y|X)) = \mathbf{E}(X)\mathbf{E}(Y)$, where the last equality is due to the assumption $\mathbf{E}(Y|X) = \mathbf{E}(Y)$.

d) Two Gaussian random variables (defined on the same probability space) are uncorrelated if and only if they are independent.

FALSE: Let X be a mean-zero Gaussian random variable and let S be a random variable which is independent of X and such that $\mathbf{P}(S = 1) = \mathbf{P}(S = -1) = 1/2$. In Exercise 3.2.11 we saw that SX is Gaussian, X and SX are uncorrelated, but X and SX are not independent.

e) If $\sigma(X) = \sigma(Y)$ for two random variables X and Y then X and Y have the same law.

FALSE: For some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ let $X = I_A$ and $Y = I_{A^c}$ for some set $A \in \mathcal{F}$ with probability $0 < \mathbf{P}(A) < 1$. Then $\sigma(X) = \{\emptyset, \Omega, A, A^c\} = \sigma(Y)$ but, for example, $\mathbf{P}(X = 1) = \mathbf{P}(A)$ and $\mathbf{P}(Y = 1) = \mathbf{P}(A^c)$, which are not equal as long as $\mathbf{P}(A) \neq 1/2$.

f) If X_n and X are random variables (which you may assume are defined on the same probability space) for which $X_n \xrightarrow{\mathcal{L}} X$ then $f(X_n) \xrightarrow{\mathcal{L}} f(X)$ for any continuous function f.

TRUE: Letting $Y_n = f(X_n)$ and $Y = f(X)$, we have by Proposition 1.4.10 that $Y_n \xrightarrow{\mathcal{L}} Y$ if and only if $\mathbf{E}(h(Y_n)) \to \mathbf{E}(h(Y))$ for all bounded, continuous functions h. Note that the composition $h(f(\cdot))$ is bounded and continuous. Therefore, since $X_n \xrightarrow{\mathcal{L}} X$ we have

$$\mathbf{E}(h(Y_n)) = \mathbf{E}(h(f(X_n)) \to \mathbf{E}(h(f(X)) = \mathbf{E}(h(Y)).$$

g) If a stochastic process $\{X_t, t \geq 0\}$ satisfies, for some finite $C > 0$, $\mathbf{E}|X_{t+h} - X_t| \leq Ch$ for all $t \geq 0$ and all $h > 0$, then $\{X_t, t \geq 0\}$ has a continuous modification.

FALSE: Consider $\Omega = [0, 1]$ with its Borel σ-field and the uniform probability measure. Let $U(\omega) = \omega$; that is, U is a Uniform(0,1) random variable. Define $X_t(\omega) = I_{\{U \le t\}}(\omega)$, $t \ge 0$. Note that $|X_{t+h} - X_t|$ is 1 if $0 \le t < 1$ and $t < U \le t + h$, and $|X_{t+h} - X_t|$ is 0 otherwise. So for $0 \le t < 1$ and any $h > 0$, $\mathbf{E}|X_{t+h} - X_t| = \mathbf{P}(t < U \le t + h) \le h$, where the inequality is due to the possibility that $t + h > 1$. So $\{X_t, t \ge 0\}$ satisfies the condition with $C = 1$, but clearly it has no continuous modification.

h) If random variables $X_n, n = 1, 2, \ldots,$ and X satisfy $\sum_{n=1}^{\infty} \mathbf{P}(|X_n - X| \ge \epsilon) < \infty$ for all $\epsilon > 0$, then X_n converges to X almost surely.

TRUE: See Grimmett & Stirzaker, Lemma 7.2.10 on page 312. Alternatively, for $\epsilon > 0$ let $A_n(\epsilon) = \{|X_n - X| \ge \epsilon\}$. Then since $\sum_{n=1}^{\infty} \mathbf{P}(A_n(\epsilon)) < \infty$ we have by the first Borel-Cantelli lemma that $\mathbf{P}(\cap_{N=1}^{\infty} \cup_{m=N}^{\infty} A_m(\epsilon)) = 0$, for any $\epsilon > 0$. Thus

$$\mathbf{P}\Big(\bigcup_{\epsilon>0,\epsilon\in\mathbb{Q}} \bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} A_m(\epsilon) \Big) \le \sum_{\epsilon>0,\epsilon\in\mathbb{Q}} \mathbf{P}\Big(\bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} A_m(\epsilon) \Big) = 0$$

Taking complements we have

$$\mathbf{P}\Big(\bigcap_{\epsilon>0,\epsilon\in\mathbb{Q}} \bigcup_{N=1}^{\infty} \bigcap_{m=N}^{\infty} \{|X_m - X| < \epsilon\} \Big) = 1.$$

Thus for each ω in the above set, which has probability 1, $X_m(\omega) \to X(\omega)$ (i.e. for all $\epsilon > 0$ there exists some $N < \infty$ such that for all $m \ge N$ we have $|X_m(\omega) - X(\omega)| < \epsilon$).

i) If $(\Omega, \mathcal{F})$ is a measurable space and $A \subseteq B \subseteq \Omega$ with $B \in \mathcal{F}$ then $A \in \mathcal{F}$.

FALSE: Consider $\Omega = \{1, 2\}$, $\mathcal{F} = \{\emptyset, \{1, 2\}\}$, $B = \{1, 2\}$, $A = \{1\}$.

j) If $X_n, n = 1, 2, \ldots$ are random variables on a probability space $(\Omega, \mathcal{F}, I\!\!P)$ which satisfy $\sup_{n\ge1} I\!\!E|X_n| < \infty$ then the collection $\{X_n, n \ge 1\}$ is uniformly integrable.

FALSE: Consider the uniform probability space with $X_n(\omega) = nI_{[0,1/n)}(\omega)$. Then $I\!\!E|X_n| = 1$ for all $n \ge 1$, so $\sup_{n\ge1} I\!\!E|X_n| = 1$. However, for any fixed M, $I\!\!E(|X_n|I_{\{|X_n|>M\}}) = 1$ for all $n > M$, and so $\lim_{M\to\infty} \sup_{n\ge1} I\!\!E(|X_n|I_{\{|X_n|>M\}}) = 1 \ne 0$, i.e. the random variables are not uniformly intergrable.