Math136/Stat219 Fall 2008
Sample Final Examination

Write your name and sign the Honor code in the blue books provided.

You have 3 hours to solve all questions, each worth points as marked (maximum of 100). Complete reasoning is required for full credit. You may cite lecture notes and homework sets, as needed, stating precisely the result you use, why and how it applies. **Important note: If you wish to use a result that is contained in an Exercise in the course notes that was not assigned for homework or proved in lecture, you must prove the result yourself (i.e. you cannot just site the Exercise number.)**

You may consult the following materials while taking the exam:

1. Stat219/Math136 Lecture notes, Fall 2008 version (the required text)
2. Kevin Ross's Lecture slides posted in Coursework
3. Homework problems and solutions posted in Coursework
4. Your own graded homework papers
5. Your own notes taken during lecture

Use of any other material is prohibited and constitutes a violation of the Honor Code. This includes, but is not limited to: other texts (including optional and recommended texts), photocopying of texts or notes, materials from previous sections of Stat219/Math136, the internet, programming formulas or other results in a calculator or computer, consultation with anyone during the exam (except for the Instructors or Teaching Assistants).

1.(5x5) Consider a simple, symmetric random walk. That is, let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables with $\mathbf{P}(\xi_k = 1) = \mathbf{P}(\xi_k = -1) = 1/2$ for all k. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} \xi_k$ for $n = 1, 2, \dots$. Let $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$.

For fixed positive integers a and b let $\tau = \min\{n \geq 0 : S_n \notin (-a, b)\}$. You may assume that $\tau < \infty$ a.s.

a) Show that τ is an $\{\mathcal{F}_n\}$-stopping time.

ANS: $\{\tau = n\} = \{S_1 \in (-a,b), \ldots, S_{n-1} \in (-a,b), S_n \notin (-a,b)\} \in \mathcal{F}_n$ since $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ (because S_n is defined through an invertible transformation of $\xi_1, \ldots, \xi_n$; see Corollary 1.2.17). This is enough to show that τ is a discrete time $\{\mathcal{F}_n\}$-stopping time.

b) Show that $\{S_n^2 - n\}$ is an $\{\mathcal{F}_n\}$-martingale.

ANS: Clearly, $\{\mathcal{F}_n\}$ is a filtration to which $\{S_n^2 - n\}$ is adapted. Checking integrability, since $\xi_k \in \{1, -1\}$, we have $|S_n| \leq n$ and so $\mathbb{E}|S_n^2 - n| \leq n^2 + n < \infty$ for any fixed n. Finally using the independence and distributions of ξ_n:

$$\begin{aligned}
\mathbb{E}[S_{n+1}^2 - (n+1)|\mathcal{F}_n] &= \mathbb{E}[(S_n + \xi_{n+1})^2|\mathcal{F}_n] - (n+1) \\
&= S_n^2 + \mathbb{E}(\xi_{n+1}^2) - 2S_n\mathbb{E}(\xi_{n+1}) - (n+1) \\
&= S_n^2 + 1 + 0 - (n+1) = S_n^2 - n
\end{aligned}$$

c) Show that $\mathbf{E}(\tau \wedge N) = \mathbf{E}(S_{\tau\wedge N}^2)$ for any fixed positive integer N.

ANS: For any fixed N, $\tau \wedge N$ is an $\{\mathcal{F}_n\}$-stopping time (see Exercise 4.3.3). Consider the stopped process $\{S_{n\wedge(\tau\wedge N)}^2 - n \wedge (\tau \wedge N)\}$. By same reasoning in part (b) we have that $|S_{n\wedge(\tau\wedge N)}^2 - n \wedge (\tau \wedge N)| \leq N^2 + N$ for all n; thus the stopped process is uniformly integrable. By the optional stopping theorem (Theorem 4.3.8),

$$\mathbb{E}[S_{\tau\wedge N}^2 - \tau \wedge N] = \mathbb{E}[S_0^2 - 0] = 0,$$

which yields the result.

d) Verify why the equality $\mathbf{E}(\tau) = \mathbf{E}(S_\tau^2)$ follows from the result in part (c).

ANS: As $N \to \infty$, $0 \leq \tau \wedge N \uparrow \tau$ a.s. Thus by the monotone convergence theorem, $\mathbb{E}[\tau \wedge N] \to \mathbb{E}[\tau]$. Using the definition of τ, we have that $S_{\tau\wedge N}^2 \leq \max(a^2, b^2)$ for all N. Also, since $\tau < \infty$ a.s., letting $N \to \infty$ we have $S_{\tau\wedge N}^2 \to S_\tau^2$ a.s. Thus by the dominated convergence theorem $\mathbb{E}[S_{\tau\wedge N}^2] \to \mathbb{E}[S_\tau^2]$. The desired equality then follows by taking limits in the result from part (c).

e) Show that $\mathbf{E}(\tau) = ab$.

ANS: Since a, b are integers and ξ_k only takes values $1, -1$, we see that $S_\tau \in \{-a, b\}$. In class we showed that $\mathbb{P}(S_\tau = -a) = b/(b+a)$. Thus using part (d) we have

$$\mathbb{E}[\tau] = \mathbb{E}[S_\tau^2] = a^2 \frac{b}{b+a} + b^2 \frac{a}{a+b} = ab.$$

2.(8x3) Parts of this problem are adapted from your homework. You should solve this problem again, and not merely cite the homework solutions. However, you can state and use results for Brownian motion without proof.

Let $\{W_t, t \geq 0\}$ be a Brownian motion with canonical filtration $\{\mathcal{F}_t^W, t \geq 0\}$. Let $Y_t = e^{W_t}$. For each of the following, verify whether or not $\{Y_t, t \geq 0\}$:

a) is a Gaussian process.

NO: For any fixed $t \geq 0$, $Y_t > 0$ a.s. so Y_t can not be a Gaussian RV and thus the process can not have Gaussian FDD's.

b) is a submartingale with respect to $\{\mathcal{F}_t^W\}$.

YES: Clearly $\{Y_t\}$ is adapted to this filtration. For fixed $t \geq 0$, $\mathbb{E}|Y_t| = e^{t/2} < \infty$ so the integrability condition is satisfied. Finally, e^x is a convex function, so by Jensen's inequality for conditional expectations and the martingale property of BM:

$$\mathbb{E}(Y_{t+h}|\mathcal{F}_t^W) = \mathbb{E}(e^{W_{t+h}}|\mathcal{F}_t^W) \geq \exp(\mathbb{E}(W_{t+h}|\mathcal{F}_t^W)) = \exp(W_t) = Y_t.$$

c) is uniformly integrable.

NO: In particular, $\mathbb{E}|Y_t| = e^{t/2}$ so $\sup_{t\geq 0} \mathbb{E}|Y_t| = \infty$ and hence $\{Y_t\}$ can not be U.I. (U.I. would imply that $\sup_{t\geq 0} \mathbb{E}|Y_t| < \infty$.)

d) is a stationary process.

NO: In particular, $\mathbb{E}(Y_t) = e^{t/2}$ which depends on t, so the process is not stationary.

e) is a homogeneous Markov process.

YES: BM is a homogeneous process and e^x is an invertible function, so e^{W_t} is also a homogeneous Markov process (result from lecture notes).

f) has continuous sample paths almost surely.

YES: Fix ω so that $t \mapsto W_t(\omega)$ is continuous. Then for such ω, $t \mapsto e^{W_t(\omega)}$ is a composition of two continuous functions and hence continuous. Since BM has continuous sample paths a.s., the set of all such ω has probability one.

g) converges almost surely as $t \to \infty$.

NO: For BM we know that $\limsup_{t\to\infty} W_t = \infty$ a.s. but $\liminf_{t\to\infty} W_t = -\infty$ a.s. Thus

$\limsup_{t\to\infty} Y_t = \infty$ a.s. but $\liminf_{t\to\infty} Y_t = 0$ a.s. So with probability one, Y_t does not converge to a limit.

h) has infinite total variation almost surely (on any time interval).

YES: Essentially follows from the fact that BM has infinite total variation almost surely on any time interval. Fix $t \in (0,\infty)$ and let $M = \inf_{0\le s\le t} e^{W_t} > 0$ a.s. Let $\pi = \{t_k\}$ be a partition of $[0,t]$. Recall that for x close to 0 we have $e^x \approx 1+x$ and that since W has continuous paths a.s. $W_{t_{k+1}} - W_{t_k}$ is close to 0 as long as $||\pi||$ is sufficiently small. So we have

$$\sum |e^{W_{t_{k+1}}} - e^{W_{t_k}}| = \sum e^{W_{t_k}} |e^{W_{t_k+1}-W_{t_k}} - 1| \ge M \sum |e^{W_{t_{k+1}}-W_{t_k}} - 1| \approx M \sum |W_{t_{k+1}} - W_{t_k}|$$

But we know that this last term approaches ∞ as $||\pi|| \to 0$ (under suitable conditions on π; see Proposition 5.3.9). So the same must be true of the LHS.

3.(3x3) A basketball team scores baskets according to a Poisson process with rate 2 baskets per minute.

a) What is the expected amount of time until the team scores its first basket?

ANS: Let $N(t)$ be the number of baskets the team has made by time t and let T_1 be the time until the first basket. Since $N(t)$ is a Poisson process, its interarrival times, and T_1 in particular, are i.i.d. exponential RV's with parameter $\lambda = 2$. Thus $\mathbb{E}T_1 = 1/2$; i.e. expected time until first basket is half minute.

b) Given that at the five minute mark of the game the team has scored exactly one basket, what is the probability that the team scored the basket in the first minute?

ANS: Conditional on the event $\{N(t) = 1\}$, T_1 is uniformly distributed on $[0,t]$ (see Proposition 6.2.8). So $\mathbb{P}(T_1 < 1|N(5) = 1) = 1/5$.

c) What is the probability that the team scores exactly three baskets in the first five minutes of the game? (Computation of the numerical value is not necessary.)

ANS: For fixed t, $N(t)$ is a Poisson RV with parameter $\lambda t = 2t$; in particular $N(5)$ is Poisson with parameter 10. Thus

$$\mathbb{P}(N(5) = 3) = \frac{e^{-10}10^3}{3!}$$

4.(14x3) State which of the following statements is true and which is false. You get 1 point for each correct answer + 2 points for its reasoning (that is, citing a specific result from lecture notes, deriving from known result or providing a counter example).

a) Any two continuous time stochastic processes with stationary and independent increments have the same finite dimensional distributions.

FALSE: Both BM and the Poission process have stationary, independent increments but clearly they do not have the same FDD's.

b) If $\{N_t\}$ is a Poisson process then $\mathbf{E}[\sin(N_{t+s} - N_t)] = \mathbf{E}[\sin(N_s)]$ for any $t, s \geq 0$.

TRUE: A Poisson process has stationary increments, so $N_{t+s} - N_t \stackrel{d}{=} N_s - N_0 = N_s$ since $N_0 = 0$. Recall that for random variables, if $X \stackrel{d}{=} Y$ then $I\!E[h(X)] = I\!E[h(Y)]$ for every bounded, Borel-measurable function h. (See Proposition 1.4.3 and following paragraph.) The conclusion in (b) follows since sin is bounded and measurable.

c) If Y is an integrable random variable, $\{\mathcal{F}_t, t \geq 0\}$ is a continuous time filtration, and $X_t = \mathbf{E}(Y|\mathcal{F}_t)$, then $\{X_t, \mathcal{F}_t\}$ is a continuous time martingale.

TRUE: By definition of conditional expectation, $I\!E(Y|\mathcal{F}_t)$ is $\mathcal{F}_t$-measurable, so $\{X_t\}$ is $\{\mathcal{F}_t\}$-adapted. By Jensen's inequality for CE, $I\!E|X_t| = I\!E|I\!E(Y|\mathcal{F}_t)| \leq I\!E(I\!E(|Y||\mathcal{F}_t)) = I\!E|Y| < \infty$. Finally, by the tower property for CE, $I\!E(X_{t+h}|\mathcal{F}_t) = I\!E(I\!E(Y|\mathcal{F}_{t+h})|\mathcal{F}_t) = I\!E(Y|\mathcal{F}_t) = X_t$.

d) If $\{(X_t, \mathcal{F}_t), t \geq 0\}$ is a martingale (on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$) with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then X_0 is equal to a constant almost surely.

TRUE: By the definition of a martingale, X_0 is $\mathcal{F}_0$-measurable and integrable. Thus $I\!E(X_0|\mathcal{F}_0) = X_0$ a.s. But also, $I\!E(X_0|\mathcal{F}_0) = I\!E(X_0)$ a.s. (Exercise 2.3.3). Since CE is unique (a.s.) we have almost surely $X_0 = I\!E(X_0)$, a finite constant.

e) A Poisson process is continuous in probability.

TRUE: Since $N_{t+h} - N_t$ has a Poisson distribution with parameter λh, we have for any $\epsilon > 0$,

$$I\!P(N_{t+h} - N_t > \epsilon) \leq I\!P(N_{t+h} - N_t \geq 1) = 1 - e^{-\lambda h},$$

which goes to 0 as $h \to 0$.

f) If $\{X_t\}$ is a supermartingale with $X_t \geq 0$ a.s. for all $t \geq 0$, then X_t converges a.s. as $t \to \infty$ to an integrable random variable.

TRUE: In this case $\{Y_t\}$ with $Y_t = -X_t$ is a submartingale with $Y_t \leq 0$ and hence $\sup_{t\geq 0} \mathbb{E}[Y_t^+] = 0 < \infty$. So by the martingale convergence theorem (Theorem 4.5.1), Y_t converges a.s. as $t \to \infty$ to some integrable RV Y_∞. Setting $X_\infty = -Y_\infty$, we see that $X_t \to X_\infty$ a.s. and X_∞ is integrable (because Y_∞ is.)

g) If $\{X_n\}$ is a discrete time martingale that converges a.s. to some integrable random variable X_∞, then X_n converges to X_∞ in L^1.

FALSE: We have seen that the critical branching process Z_n ($m = 1$ and $\mathbb{P}(N = 1) < 1$) converges to 0 a.s. (Proposition 4.6.5). However, $\mathbb{E}(Z_n) = 1$ for all n and so Z_n does not converge to 0 in L^1.

h) If $\{X_t, t \geq 0\}$ is a Markov process with state space $\mathbf{S}$ then for every fixed $t, h \geq 0$ and each bounded measurable function $f : \mathbf{S} \to \mathbb{R}$ there is some Borel-measurable function $g : \mathbf{S} \to \mathbb{R}$ (which may depend on t and h) for which $\mathbf{E}[f(X_{t+h})|\sigma(X_s : 0 \leq s \leq t)] = g(X_t)$.

TRUE: Since $\{X_t\}$ is Markov, $\mathbf{E}[f(X_{t+h})|\sigma(X_s : 0 \leq s \leq t)] = \mathbf{E}[f(X_{t+h})|\sigma(X_t)]$ for any bounded, measurable function f. By definition of CE, $\mathbf{E}[f(X_{t+h})|\sigma(X_t)]$ is a $\sigma(X_t)$-measurable RV. But any $\sigma(X_t)$-measurable RV is given by $g(X_t)$ for some Borel measurable function g (Theorem 1.2.16).

i) The Poisson process N_t is a stationary stochastic process.

FALSE: For example $\mathbf{E}N_t = \lambda t$, which depends on t.

j) Any super-martingale is also a martingale.

FALSE: Take $X_n = -n$. Easy to check that this is a super-martingale but not a martingale.

k) If $\mathbf{E}|X_n| \to 0$ as $n \to \infty$ then also $\mathbf{E}X_n^2 \to 0$ as $n \to \infty$.

FALSE: Take $X_n(\omega) = nI_{[0,n^{-2}]}(\omega)$ defined on the uniform probability space on $[0, 1]$. Then $\mathbf{E}|X_n| = n^{-1} \to 0$ while $\mathbf{E}X_n^2 = 1$ for all n.

l) If a Gaussian stochastic process $\{X_t, t \geq 0\}$ has almost surely continuous sample paths and is a martingale with respect to its canonical filtration then $\lim_{t\to\infty} X_t$ exists w.p.1. and is integrable.

FALSE: The Brownian motion is such a process but we have $\limsup_{t\to\infty} W_t = \infty$ a.s. and $\liminf_{t\to\infty} W_t = -\infty$ a.s.

m) If $\{W_t, t \geq 0\}$ is a Brownian motion and $\tau = \inf\{t \geq 0 : W_t = 1\}$ then $\{W_{t\wedge\tau}, t \geq 0\}$ is a Brownian motion.

FALSE: We have seen that $\tau < \infty$ a.s. (see Section 5.2 before equation (5.2.1).) Thus $t \wedge \tau \to \tau$ a.s. as $t \to \infty$ and so $\limsup_{t\to\infty} W_{t\wedge\tau} = W_\tau = 1$ a.s. But for Brownian motion we know that $\limsup_{t\to\infty} W_t = \infty$ a.s.

n) The process $\{A_t\}$ in the Doob-Meyer decomposition of $\{e^{W_t - t/2}\}$ (where $\{W_t\}$ is a Brownian motion) satisfies $A_t = e^t - 1, t \geq 0$.

FALSE: Let $X_t = e^{W_t - t/2}$. The process $\{A_t\}$ is such that, in particular, $\{X_t^2 - A_t\}$ is a martingale. But for $0 \leq s \leq t$, using independent increments of Brownian motion, we have

$$\mathbb{E}[X_t^2 - (e^t - 1)|\mathcal{F}_s] = \mathbb{E}[e^{2W_t - t}|\mathcal{F}_s] - e^t + 1 = e^{2W_s - t}\mathbb{E}[e^{2(W_t - W_s)}] - e^t + 1 = e^{2W_s - t + 2(t-s)} - e^t + 1.$$

In general, this is not equal to $X_s^2 - (e^s - 1) = e^{2W_s - s} - e^s + 1$ (e.g. take $t = 3, s = 1$.) Thus, $X_t^2 - (e^t - 1)$ is not a martingale and so A_t is not given by $e^t - 1$. (In Exercise 4.4.10 you showed that $A_t = \int_0^t e^{2W_s - s} ds$.)