

**Math136/Stat219 Fall 2008**  
**Midterm Examination**  
**Friday, October 24, 2008, 11:00am - 12:30pm**

**Write your name and sign the Honor code in the blue books provided.**

You have 90 minutes to solve all questions, each worth points as marked (maximum of 50). Complete reasoning is required for full credit. You may cite lecture notes and homework sets, as needed, stating precisely the result you use, why and how it applies.

You may consult the following materials while taking the exam:

1. Stat219/Math136 Lecture notes, Fall 2008 version (the required text)
2. Kevin Ross's Lecture slides posted in Coursework (Fall 2008 only)
3. Homework problems, sample exams, and solutions posted in Coursework (Fall 2008 only)
4. Your own graded homework papers
5. Your own notes, handwritten or typed

**Use of any other material is prohibited and constitutes a violation of the Honor Code.** This includes, but is not limited to: other texts (including optional and recommended texts), photocopying of texts or notes, materials from previous sections of Stat219/Math136, the internet, programming formulas or other results in a calculator or computer, consultation with anyone during the exam (except for the Teaching Assistants or the Instructor).

- 1.** (3 Points each) On a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , let  $Y$  be a random variable with  $\mathbf{E}(Y^2) < \infty$  and  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -field. Define

$$\text{Var}(Y|\mathcal{G}) = \mathbf{E}(Y^2|\mathcal{G}) - (\mathbf{E}(Y|\mathcal{G}))^2.$$

Show the following. (**Note: you must give a proof. Merely citing Exercise 2.3.7 will receive no credit.**)

- a) Show that if  $Y$  is  $\mathcal{G}$ -measurable then  $\text{Var}(Y|\mathcal{G}) = 0$  almost surely.

**ANS.** Since  $Y$  is  $\mathcal{G}$ -measurable,

$$\text{Var}(Y|\mathcal{G}) = \mathbf{E}(Y^2|\mathcal{G}) - (\mathbf{E}(Y|\mathcal{G}))^2 = Y^2 - (Y)^2 = 0.$$

b) Show that

$$\text{Var}(Y) = \mathbf{E}(\text{Var}(Y|\mathcal{G})) + \text{Var}(\mathbf{E}(Y|\mathcal{G})).$$

**ANS.** Using the tower property and linearity of CE gives

$$\mathbf{E}(\text{Var}(Y|\mathcal{G})) = \mathbf{E}[\mathbf{E}(Y^2|\mathcal{G}) - (\mathbf{E}(Y|\mathcal{G}))^2] = \mathbf{E}(Y^2) - \mathbf{E}[\mathbf{E}(Y|\mathcal{G})^2].$$

The definition of (unconditional) variance and the tower property implies

$$\text{Var}(\mathbf{E}(Y|\mathcal{G})) = \mathbf{E}[(\mathbf{E}(Y|\mathcal{G}))^2] - (\mathbf{E}[\mathbf{E}(Y|\mathcal{G})])^2 = \mathbf{E}[(\mathbf{E}(Y|\mathcal{G}))^2] - (\mathbf{E}[Y])^2.$$

Adding the above two equations yields

$$\mathbf{E}(\text{Var}(Y|\mathcal{G})) + \text{Var}(\mathbf{E}(Y|\mathcal{G})) = \mathbf{E}(Y^2) - (\mathbf{E}(Y))^2 = \text{Var}(Y).$$

c) Suppose that  $Y$  is  $\mathcal{G}$ -measurable and  $X$  is a random variable on  $(\Omega, \mathcal{F}, \mathbf{P})$  with  $\mathbf{E}(X^2) < \infty$ . Show that

$$\text{Var}(XY|\mathcal{G}) = Y^2\text{Var}(X|\mathcal{G}).$$

**ANS.** Since  $Y$  is  $\mathcal{G}$ -measurable, taking out what is known yields

$$\text{Var}(XY|\mathcal{G}) = \mathbf{E}((XY)^2|\mathcal{G}) - (\mathbf{E}(XY|\mathcal{G}))^2 = Y^2\mathbf{E}(X^2|\mathcal{G}) - (Y\mathbf{E}(X|\mathcal{G}))^2 = Y^2\text{Var}(X|\mathcal{G}).$$

**2.** (3 Points each) Consider the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , where  $\Omega = (0, 1)$ ;  $\mathcal{F}$  is the Borel  $\sigma$ -field on  $(0, 1)$ , that is,  $\mathcal{F} = \sigma(\{(a, b) : 0 < a < b < 1\})$ ; and  $\mathbf{P}$  is the uniform probability measure. For  $n = 1, 2, \dots$  define  $X_n(\omega) = 2^n I_{A_n}(\omega)$ ,  $\omega \in \Omega$ , where

$$A_n = \left( \frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} + \frac{1}{2^n} \right).$$

a) Does  $X_n$  converge to a random variable  $X$  almost surely as  $n \rightarrow \infty$ ? If so, specify  $X$  and prove the convergence. If not, state why not.

**ANS.**  $X_n$  converges to 0 almost surely. To see this, fix  $\omega \in (0, 1) \setminus \{1/2\}$ . Choose  $N$  large enough so that  $\omega \in A_N^c$ , which implies  $\omega \in A_n^c$  for all  $n \geq N$ . Thus  $X_n(\omega) = 0$  if  $n \geq N$  and so  $X_n(\omega)$  converges to 0 for this particular  $\omega$ . Since the set  $\omega \in (0, 1) \setminus \{1/2\}$  has probability 1, and  $X_n(\omega)$  converges to 0 for any  $\omega$  in this set, we have  $X_n \rightarrow 0$  almost surely.

**b)** For which values of  $q \geq 1$ , if any, does  $X_n$  converge to a random variable  $X$  in  $L^q$  as  $n \rightarrow \infty$ ?

**ANS.** By part (a) and Proposition 1.3.25,  $X = 0$  a.s. is the only candidate for the  $L^q$  limit. However, with  $q = 1$ , we have for any  $n$ ,

$$\mathbf{E}|X_n - 0| = \mathbf{E}(2^n I_{A_n}) = 2^n \mathbf{P}(A_n) = 2^n (2/2^n) = 2.$$

Thus  $\mathbf{E}|X_n - 0|$  does not converge to 0 as  $n \rightarrow \infty$ , and so  $X_n$  does not converge in  $L^1$ . It follows from Corollary 1.3.19 that  $X_n$  does not converge in  $L^q$  for any value of  $q \geq 1$ .

**c)** Specify  $\sigma(X_1, X_2, \dots, X_n)$ . (Your answer can be of the form “ $\sigma(X_1, X_2, \dots, X_n)$  contains all sets of the form...” As long as you describe the  $\sigma$ -field properly in this manner, there is no need for further proof.)

**ANS.**  $\sigma(X_1, X_2, \dots, X_n)$  contains the set  $A_n$ , all sets of the form

$$\left(\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}}\right) \cup \left(\frac{1}{2} + \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^k}\right)$$

for  $k = 1, 2, \dots, n - 1$ , and all countable union of the above sets.

*Note:*  $\left(\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}}\right) \notin \sigma(X_1, X_2, \dots, X_n)$ .

**3.** (4 Points each) State which of the following statements is true and which is false. You get 1 point for each correct TRUE/FALSE answer + 3 points for its reasoning (that is, citing a specific result from lecture notes, deriving from known result or providing a counter example). You may cite any Exercises, Theorems, etc from Chapters 1 through 3 of the text, without proof, as you long you explain why and how it applies.

**a)** Suppose that  $\{X_t, 0 \leq t \leq T\}$  is a square integrable stochastic process which satisfies for some  $H \in (1/2, 1)$ ,

$$\mathbf{E}(X_t X_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) \quad , \quad s, t \in [0, T].$$

True or false:  $\{X_t, t \geq 0\}$  has a continuous modification.

**TRUE.** Existence of a continuous modification follows from Kolmogorov's continuity criteria with  $\alpha = 2$ ,  $\beta = 2H - 1 > 0$  (since  $H > 1/2$ ), and  $c = 1$ .

$$\mathbf{E}[|X_t - X_s|^2] = \mathbf{E}(X_t^2) + \mathbf{E}(X_s^2) - 2\mathbf{E}(X_t X_s) = t^{2H} + s^{2H} - 2\frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) = |t - s|^{2H}.$$

b) On some probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  let  $A_1, A_2, \dots \in \mathcal{F}$  be such that  $\mathbf{P}(A_i) = 1$ ,  $i = 1, 2, \dots$ . True or false:  $\mathbf{P}(\bigcap_{i=1}^{\infty} A_i) = 1$ .

**TRUE.** Note that  $\mathbf{P}(A_i^c) = 0$  for all  $i$ , and therefore countable sub-additivity implies

$$0 \leq \mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) \leq \sum_{i=1}^{\infty} \mathbf{P}(A_i^c) = 0.$$

Then

$$\mathbf{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - \mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 1.$$

c) Suppose that a stochastic process  $\{X_t, t \geq 0\}$  is a version of  $\{Y_t, t \geq 0\}$ . True or false: for any  $T > 0$ ,  $\inf_{0 \leq t \leq T} X_t$  has the same law as  $\inf_{0 \leq t \leq T} Y_t$ .

**FALSE.** Recalling Example 3.1.7, on the uniform probability space on  $(0, 1)$  let  $Y_t \equiv 0$  and  $X_t(\omega) = -1_{\{t\}}(\omega)$ . Then  $\mathbf{P}(\inf_{0 \leq t \leq 1} Y_t = 0) = 1$  but  $\mathbf{P}(\inf_{0 \leq t \leq 1} X_t = -1) = 1$ .

d) Suppose that  $X_n$  is an integrable RV for each  $n = 1, 2, \dots$ . True or false:  $\mathbf{E}(\sum_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} \mathbf{E}(X_n)$ .

**FALSE.** Let  $X_1$  satisfy  $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$  and let  $X_n = X_1$  for all  $n$ . Then each  $X_n$  is integrable and  $\sum_{n=1}^{\infty} \mathbf{E}(X_n) = 0$ . However, letting  $Y = \sum_{n=1}^{\infty} X_n$ , we have  $\mathbf{P}(Y = \infty) = \mathbf{P}(Y = -\infty) = 1/2$ , and thus  $\mathbf{E}(Y)$  is not defined.

*This is similar to Exercise 1.4.30, in that both are really questions about convergence of expectations. In that exercise, the variables were nonnegative so you could apply monotone convergence. For convergence of expectations in general, you need some kind of uniform integrability condition (e.g. dominated convergence theorem) to interchange limits and expectation. The absence of the uniform integrability in problem d should have suggested the answer was false. (See part (h) for more comments on uniform integrability.)*

e) Suppose that  $\{X_n, n = 0, 1, \dots\}$  is a stochastic process such that  $X_n$  has the same law as  $X_0$  for all  $n$ . True or false:  $\{X_n\}$  is a stationary stochastic process.

**FALSE.** This essentially follows from the observation that a random vector  $(X, Y)$  need not have the same law as  $(U, V)$  even if  $X$  has the same law as  $U$  and  $Y$  has the same law as  $V$ . To see this, let  $X$  be a RV with  $\mathbf{P}(X = 1) = \mathbf{P}(X = -1) = 1/2$  and let  $Y = U = X$  and  $V = -X$ . Then  $X$  has the same law as  $U$ ,  $Y$  has the same law as  $V$ , but  $\mathbf{P}(X = 1, Y = 1) = \mathbf{P}(X = 1) = 1/2$  and  $\mathbf{P}(U = 1, V = 1) = 0$ . The claim in the problem is false since stationarity is a property of *joint* distributions and the assumption only determines distributional properties of the marginal distributions. (For a continuous time counterexample, let  $X_0 \sim N(0, 1)$  and  $X_t = W_t/\sqrt{t}, t > 0$  where  $W$  is a Brownian motion. Then  $X_t \sim N(0, 1)$  for all  $t \geq 0$ , but for  $t > s > 0$ ,  $\mathbf{E}(X_t X_s) = \mathbf{E}(W_t W_s)/\sqrt{ts} = \sqrt{s/t}$ , which depends on  $t, s$  individually, and not just through their difference  $|t - s|$ .)

*Note:  $\{X_n\}$  is a stationary process is NOT equivalent to “ $X_n$  has the same distribution for all  $n$ ”, as the above example shows.*

f) On a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  let  $A, B \in \mathcal{F}$  with  $A \subset B$ . True or false:  $\mathbf{E}(I_B|I_A) = 1$  almost surely.

**FALSE.** Consider  $A^c \in \sigma(I_A)$ . Then  $\mathbf{E}(1I_{A^c}) = \mathbf{P}(A^c)$  but  $\mathbf{E}(I_B I_{A^c}) = \mathbf{P}(B \cap A^c)$ . Since in general  $\mathbf{P}(A^c) \neq \mathbf{P}(B \cap A^c)$ , the partial averaging condition fails to hold for the set  $A^c$  and thus the constant 1 is not a version of  $\mathbf{E}(I_B|I_A)$ . (You can check that in this case  $\mathbf{E}(I_B|I_A) = I_A + \mathbf{P}(B|A^c)I_{A^c}$  which is not 1 as long as  $\mathbf{P}(B|A^c) < 1$ .)

g) Let  $\{X_t, t \geq 0\}$  be a stochastic process which satisfies  $\mathbf{E}|X_t - X_s| \leq |t - s|$  for all  $s, t \geq 0$ . True or false: for each  $t \geq 0$ ,  $X_{t+1/n}$  converges to  $X_t$  in probability as  $n \rightarrow \infty$ .

**TRUE.** Fix  $t \geq 0$ . For  $\epsilon > 0$ , Markov's inequality yields

$$\mathbf{P}(|X_{t+1/n} - X_t| \geq \epsilon) \leq \mathbf{E}|X_{t+1/n} - X_t|/\epsilon,$$

and the RHS converges to 0 as  $n \rightarrow \infty$  by assumption. The LHS converges to 0 for any  $\epsilon > 0$  which is just the definition of convergence in probability of  $X_{t+1/n}$  to  $X_t$ .

*In this problem you can also check that  $X_{t+1/n}$  converges to  $X_t$  in  $L^1$  as  $n \rightarrow \infty$ , which implies convergence in probability.*

h) Let  $\{X_n, n = 0, 1, 2, \dots\}$  be a stationary stochastic process with  $\mathbf{E}(X_n^2) < \infty$  for all  $n$ .

True or false:  $\{X_n, n = 0, 1, 2, \dots\}$  is a uniformly integrable collection of random variables.

**TRUE.** By stationarity we have for all  $n \geq 0$ ,  $\mathbf{E}|X_n|^2 = \mathbf{E}|X_0|^2 = c < \infty$ . Therefore  $\sup_{n \geq 0} \mathbf{E}|X_n|^2 = c < \infty$  and so the process is uniformly integrable (see Exercise 1.4.24 with  $f(x) = x^2$ .)

Alternatively, since the process is stationary,  $X_n$  has the same law as  $X_0$  and you can use the change of variables formula (Proposition 1.4.3) to check that this implies  $\mathbf{E}[|X_n| 1_{\{|X_n|>M\}}] = \mathbf{E}[|X_0| 1_{\{|X_0|>M\}}]$  for any  $M$ . Therefore

$$\lim_{M \rightarrow \infty} \sup_n \mathbf{E}[|X_n| 1_{\{|X_n|>M\}}] = \lim_{M \rightarrow \infty} \mathbf{E}[|X_0| 1_{\{|X_0|>M\}}] = 0,$$

where the last equality is due to Exercise 1.2.27. Thus, by definition the random variables are uniformly integrable.

*There was a lot of confusion on this problem. In Lemma 1.4.26,  $\sup_n \mathbf{E}|X_n| < \infty$  is not enough to imply uniform integrability; you need part (b) there to be true also. Even so,  $\mathbf{E}|X_n| < \infty$  for all  $n$  does not imply  $\sup_n \mathbf{E}|X_n| < \infty$ . (Just think about real numbers:  $x_n < \infty$  for all  $n = 0, 1, 2, \dots$  does not imply  $\sup_n x_n < \infty$ ; just take  $x_n = n$ .) Example 1.4.24 gives some sufficient conditions for U.I. In particular, U.I. is really only a question when you have an infinite collection of random variables. Again, having each  $X_n$  integrable does not imply U.I. for an infinite collection of random variables. It follows from Example 1.4.24 that  $\sup_n \mathbf{E}|X_n|^{1+\epsilon} < \infty$  for some  $\epsilon > 0$  implies U.I. As Lemma 1.4.26 shows, two things are needed for U.I. — a uniform bound on the expected values and a uniform rate at which the “tail probabilities” converge to 0. The key in part (h) which gives the uniform bounds is the assumption that the process is stationary.*

General note on counter-examples. When describing counter-examples, you need to be as precise as possible, for instance specifying the probability space on which the RVs are defined. Many people made statements like “let  $X_n = n 1_{[0,1/n]}$ , therefore  $\mathbf{E}(X_n) = 1$ ” without making any reference to the uniform probability space.