Last Time

- Definition of conditional expectation, general case
- $\mathbb{E}(X|\mathcal{G})$, is the (a.s.) unique **random variable** which satisfies:
  - *Measurability*: $\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable
  - *Partial averaging*: for all sets $G \in \mathcal{G}$,

$$\mathbb{E}[(X - \mathbb{E}(X|\mathcal{G}))I_G] = 0$$

- Special cases of CE

Today's lecture: Sections 2.3, 2.4

Basic Properties of CE

- If $X = c$ a.s. for $c \in \mathbb{R}$ then $\mathbb{E}(X|\mathcal{G}) = c$ a.s.
- If X is integrable and $X \geq 0$ a.s. then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.
- *Monotonicity*: if X and Y are integrable and $X \geq Y$ a.s. then

$$\mathbb{E}(X|\mathcal{G}) \geq \mathbb{E}(Y|\mathcal{G}) \text{ a.s.}$$

- *Linearity*: if X and Y are integrable then for all $\alpha, \beta \in \mathbb{R}$,

$$\mathbb{E}(\alpha X + \beta Y|\mathcal{G}) = \alpha \mathbb{E}(X|\mathcal{G}) + \beta \mathbb{E}(Y|\mathcal{G}) \text{ a.s.}$$

- If X is integrable then

$$|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X||\mathcal{G})$$

Convergence of CE's

- *Monotone convergence for CE*: If $0 \le X_n \uparrow X$ a.s. and $\mathbb{E}(X) < \infty$ then

$$\mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G}) \text{ a.s.}$$

- *Fatou's lemma for CE*: If $0 \le X_n$ and X_n are integrable then

$$\mathbb{E}(\liminf_{n\to\infty} X_n|\mathcal{G}) \le \liminf_{n\to\infty} \mathbb{E}(X_n|\mathcal{G}) \text{ a.s.}$$

- *Dominated convergence for CE*: If $|X_n| \le Y$ a.s. for some integrable Y and $X_n \to X$ a.s. then

$$\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G}) \text{ a.s.}$$

Jensen's Inequality for CE

- If $g : I\!R \to I\!R$ is convex and X and $g(X)$ are integrable then

$$g(I\!E(X|\mathcal{G})) \le I\!E(g(X)|\mathcal{G}) \text{ a.s.}$$

Tower Property of CE

- If X is integrable and $\mathcal{G}_1$ and $\mathcal{G}_2$ are σ-fields with $\mathcal{G}_1 \subset \mathcal{G}_2$ then

$$\begin{aligned} \mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) &= \mathbb{E}(X|\mathcal{G}_1) \\ &= \mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) \end{aligned}$$

- “Law of iterated expectations” or “smallest σ-field wins”
- Special case: consider $\mathcal{G}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$; then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$$

"Taking out what is known"

- If Y is $\mathcal{G}$-measurable and X and XY are integrable then

$$\mathbb{E}(XY|\mathcal{G}) = Y\,\mathbb{E}(X|\mathcal{G}) \text{ a.s.}$$

Regular Conditional Probability Distribution

Let X be a RV on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Let $\mathcal{B}$ be the Borel σ-field on $\mathbb{R}$.
The function $\hat{\mathbb{P}}(B|\mathcal{G}) : (\Omega, \mathcal{B}) \mapsto [0,1]$ is called the **regular conditional probability distribution of X given $\mathcal{G}$** (RCPD) if:

- For each fixed $B \in \mathcal{B}$, $\hat{\mathbb{P}}(B|\mathcal{G}) = \mathbb{E}(I_{\{X \in B\}}|\mathcal{G})$. That is
  - $\hat{\mathbb{P}}(B|\mathcal{G})$ is $\mathcal{G}$-measurable
  - For any $A \in \mathcal{G}$, $\mathbb{P}(A \cap \{X \in B\}) = \mathbb{E}(I_A \hat{\mathbb{P}}(B|\mathcal{G}))$
- For each fixed $\omega \in \Omega$, $\hat{\mathbb{P}}(\cdot|\mathcal{G})$ is a probability measure on $(\mathbb{R}, \mathcal{B})$

For any RV X and any σ-field $\mathcal{G}$ there exists a RCPD.
Alternate notation: $\hat{\mathbb{P}}_{X|\mathcal{G}}(B, \omega)$

RCPD and Conditional Expectation

The conditional expectation of an integrable RV X given a σ-field $\mathcal{G}$ can be expressed as

$$\mathbb{E}(X|\mathcal{G}) = \int_{\mathbb{R}} x \hat{\mathbb{P}}_{X|\mathcal{G}}(dx, \omega)$$

The RHS is to be interpreted as an expected value on the probability space $(\mathbb{R}, \mathcal{B}, \hat{\mathbb{P}}_{X|\mathcal{G}}(\cdot, \omega))$, for each fixed ω.

RCPD, Conditional Expectation, and PDFs

- Suppose the random vector (X,Y) has a probability density function $f_{X,Y}$: for all $x,y \in \mathbb{R}$

$$\mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dudv$$

- The RCPD of X given $\sigma(Y)$ has has a density function

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

  where the marginal density of Y is: $f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(u,y)du$

- The conditional expectation of X given $\sigma(Y)$ is

$$\mathbb{E}(X|Y) = \int_{\mathbb{R}} x f_{X|Y}(x|Y)dx$$