Last Time

- Definition of conditional expectation in L^2 case
- $\mathbb{E}(X|Y)$, is the (a.s.) unique **random variable** which satisfies:
  - *Measurability*: $\mathbb{E}(X|Y) \in L^2(\Omega, \sigma(Y), \mathbb{P})$
  - *Orthogonality*: For all $V \in L^2(\Omega, \sigma(Y), \mathbb{P})$,

$$\mathbb{E}[(X - \mathbb{E}(X|Y))V] = 0$$

- Hilbert space properties of L^2

Today's lecture: Sections 2.1, 2.3

Definition of CE: General Case

- Let X be an integrable RV on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a σ-field with $\mathcal{G} \subset \mathcal{F}$
- The **condition expectation of X given $\mathcal{G}$** is the (a.s.) unique RV $\mathbb{E}(X|\mathcal{G})$ which satisfies:
  - *Measurability*: $\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable
  - *Partial averaging*: for all sets $G \in \mathcal{G}$,

$$\mathbb{E}[(X - \mathbb{E}(X|\mathcal{G}))I_G] = 0$$

- The partial averaging condition is equivalent to:

$$\mathbb{E}[(X - \mathbb{E}(X|\mathcal{G}))V] = 0$$

for all bounded, $\mathcal{G}$-measurable RV's V

Special Cases of CE

- If X is integrable and $\mathcal{G}$-measurable then

$$\mathbb{E}(X|\mathcal{G}) = X \text{ a.s.}$$

- Let $\mathcal{F}_0 \doteq \{\emptyset, \Omega\}$. Then for any integrable RV X,

$$\mathbb{E}(X|\mathcal{F}_0) = \mathbb{E}(X) \text{ a.s.}$$

- If X is an integrable RV independent of $\mathcal{G}$, then

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X) \text{ a.s.}$$

Versions of CE

- CE is almost surely unique: if $\mathbb{E}(X|\mathcal{G})$ and $\tilde{\mathbb{E}}(X|\mathcal{G})$ both satisfy the measurability and partial averaging conditions, then $\mathbb{E}(X|\mathcal{G}) = \tilde{\mathbb{E}}(X|\mathcal{G})$ a.s.
- We say that these RV's are *versions* of the CE.
- Notation: write $\mathbb{E}(X|Y)$ for $\mathbb{E}(X|\sigma(Y))$ and $\mathbb{E}(X|Y_1, \ldots, Y_n)$ for $\mathbb{E}(X|\sigma(Y_1, \ldots, Y_n))$

Partial Averaging Condition

- Suppose X and Y take finitely many values
- The sets $\{Y = y_i\}$ create a disjoint partition of Ω
- On $\{Y = y_i\}$, $\mathbb{E}(X|Y)$ equals the constant $\mathbb{E}(X|Y = y_i)$

$$
\begin{aligned}
& \int_{\{Y=y_i\}} \mathbb{E}(X|Y) d\mathbb{P} = \mathbb{E}(X|Y = y_i)\mathbb{P}(Y = y_i) \\
= \; & \sum_x x\mathbb{P}(X = x|Y = y_i)\mathbb{P}(Y = y_i) = \sum_x x\mathbb{P}(X = x, Y = y_i) \\
= \; & \sum_x \sum_y x I_{\{y=y_i\}} \mathbb{P}(X = x, Y = y) = \int_\Omega X I_{\{Y=y_i\}} d\mathbb{P} \\
= \; & \int_{\{Y=y_i\}} X d\mathbb{P}
\end{aligned}
$$

Partial Averaging Condition (cont.)

- Since Y takes finitely many distinct values $y_1, \dots, y_n$, the sets $\{Y = y_i\}$ are disjoint, $\cup_{i=1}^n \{Y = y_i\} = \Omega$, and therefore $\sigma(Y) = \{$all unions of $\{\omega : Y(\omega) = y_i\}\}$
- Thus for any set $G \in \sigma(Y)$, G is a union of disjoint sets $\{Y = y_k\}$ and so

$$I_G = \sum_k I_{\{Y = y_k\}}$$

- Using linearity of expectation, for all $G \in \sigma(Y)$,

$$\int_G \mathbb{E}(X|Y) d\mathbb{P} = \int_G X d\mathbb{P}$$

Existence of CE

- Theorem 2.1.6 states that for an integrable RV X and a σ-field $\mathcal{G}$, the CE $\mathbb{E}(X|\mathcal{G})$ exists and is a.s. unique
- Key steps in proof:
- Assume $X \geq 0$ a.s.
- Consider the *truncated* RV's $X_n \doteq X \wedge n$
- $X_n \in L^2$ so $\mathbb{E}(X_n|\mathcal{G})$ exists and is a.s. unique
- Define $\mathbb{E}(X|\mathcal{G}) \doteq \lim_{n\to\infty} \mathbb{E}(X_n|\mathcal{G})$ and check that this is well-defined
- Use monotone convergence theorem to show that the partial averaging condition holds for $\mathbb{E}(X|\mathcal{G})$
- For an integrable RV X, define $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X_+|\mathcal{G}) - \mathbb{E}(X_-|\mathcal{G})$