Last Time

- L^q space
- Convergence in L^q
- Weak convergence
- Uniform integrability

Today's lecture: Sections 1.4.2, 1.4.3

Uniform Integrability and Convergence

- If $X_n \to X$ in probability and $\{|X_n|^q\}$ is U.I. then $X \in L^q$. More precisely:
    - Let $X, X_1, X_2, \ldots$ be RV's on the same probability space.
    - If $X_n \to X$ in probability and $\{|X_n|^q\}$ is U.I. then $X \in L^q$ and $X_n \to X$ in L^q.
- If $X_n \xrightarrow{\mathcal{L}} X$ and $\{|X_n|^q\}$ is U.I. then $X_n \to X$ in L^q. More precisely:
    - Let $X, X_1, X_2, \ldots$ be RV's, possibly defined on different probability spaces.
    - If $X_n \xrightarrow{\mathcal{L}} X$ and $\{|X_n|^q\}$ is U.I. then $X \in L^q$ and $X_n \to X$ in L^q.

Convergence of Expectations

- Let $X, Y, X_1, X_2, \ldots$ be RV's defined on $(\Omega, \mathcal{F}, \mathbb{P})$
- **Monotone Convergence Theorem:** If $0 \le X_n \uparrow X$ a.s. then $\mathbb{E}(X_n) \uparrow \mathbb{E}(X)$.
- **Fatou's Lemma:** If there exists a RV $X \in L^1$ such that $X_n \ge X$ a.s. for all n,

$$\liminf_{n \to \infty} \mathbb{E}(X_n) \ge \mathbb{E}(\liminf_{n \to \infty} X_n)$$

- **Dominated Convergence Theorem:** If $X_n \to X$ in probability and there exists a RV $Y \in L^1$ such that $|X_n| \le Y$ a.s. for all n, then $\mathbb{E}(X_n) \to \mathbb{E}(X)$.

Independence of Events

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Two events $A, B \in \mathcal{F}$ are **$\mathbb{P}$-independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

- Events $A_1, A_2, \ldots \in \mathcal{F}$ are $\mathbb{P}$-independent if for all n and distinct indices $i_1, i_2, \ldots, i_n$

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}) = \prod_{k=1}^{n} \mathbb{P}(A_{i_k})$$

Independence of σ-fields

- Two σ-fields $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{F}$ are **$\mathbb{P}$-independent** if for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$$

- σ-fields $\mathcal{A}_1, \mathcal{A}_2, \ldots \subset \mathcal{F}$ are $\mathbb{P}$-independent if for all n, distinct indices $i_1, i_2, \ldots i_n$, and every choice of $A_1 \in \mathcal{A}_{i_1}, A_2 \in \mathcal{A}_{i_2}, \ldots, A_n \in \mathcal{A}_{i_n}$ the events $A_1, A_2, \ldots, A_n$ are $\mathbb{P}$-independent

Independence of Random Variables

- Two RV's X and Y are **$\mathbb{P}$-independent** if their generated σ-fields $\sigma(X)$ and $\sigma(Y)$ are $\mathbb{P}$-independent
  - Independence of two random vectors $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ is defined similarly
- Random variables (or vectors) $X_1, X_2, \ldots$ are $\mathbb{P}$-independent if their generated σ-fields $\sigma(X_1), \sigma(X_2), \ldots$ are $\mathbb{P}$-independent
- Functions of independent RV's are independent, i.e.
  - If $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ are independent random vectors and that $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ are Borel-measurable functions
  - Then $h(X_1, \ldots, X_n)$ and $g(Y_1, \ldots, Y_n)$ are independent random variables

Equivalent Conditions for Independence of RV's

Two RV's X and Y are independent if and only if

- Joint distribution function factors: for all $x, y \in \mathbb{R}$

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

- Joint law factors: for all Borel sets $A, B \in \mathcal{B}$

$$\mathbb{P}_{X,Y}(A \times B) = \mathbb{P}_X(A)\mathbb{P}_Y(B)$$

- Joint density factors (provided (X,Y) has a joint density): for all $x, y \in \mathbb{R}$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Similar conditions exist for more than two RV's

Independence and Expectation

- Square integrable RV's X and Y are **uncorrelated** if

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

- Independent RV's are uncorrelated: if X and Y are independent, integrable RV's then XY is integrable and

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

- But uncorrelated RV's are not necessarily independent

Independence and Expectation

- Two random vectors $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_m)$ are independent if and only if

  $$\begin{aligned} & \mathbb{E}(h(X_1, \dots, X_n)g(Y_1, \dots, Y_m)) \\ = \quad & \mathbb{E}(h(X_1, \dots, X_n))\mathbb{E}(g(Y_1, \dots, Y_m)) \end{aligned}$$

  for all bounded, Borel-measurable functions $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$