Last Time

- Definition and properties of expectation
- Integrability
- Inequalities (Jensen, Markov, Schwarz)

Today's lecture: Sections 1.3.1, 1.4.1

Law of a RV

- Let X be a RV on $(\Omega, \mathcal{F}, \mathbb{P})$. The **law (aka distribution)** of X is the probability measure $\mathbb{P}_X$ on $(\mathbb{R}, \mathcal{B})$ defined for $B \in \mathcal{B}$ as

$$\mathbb{P}_X(B) \doteq \mathbb{P}(X \in B)$$

- $\mathbb{P}_X$ determines the values $\mathbb{P}$ on $\sigma(X)$

Distribution Function of a RV

- Let X be a RV on $(\Omega, \mathcal{F}, I\!\!P)$. The **(cumulative) distribution function** of X is the function $F_X : I\!\!R \to [0,1]$ defined for $x \in I\!\!R$ as

$$F_X(x) \doteq I\!\!P(X \leq x) = I\!\!P_X((-\infty, x])$$

- The distribution function F_X uniquely determines the law $I\!\!P_X$ of X
- Any RV has a distribution function
- Special cases:
  - Discrete RV: $F_X(x) = \sum_{i:x_i \leq x} I\!\!P(X = x_i)$
  - Abs cont RV: $F_X(x) = \int_{-\infty}^{x} f_X(u) du$. F_X is continuous and $\frac{dF_X}{dx}(x)$ exists and equals $f_X(x)$ for almost every x

Properties of Distribution Function

- Let X be a RV with distribution function F_X. Then
  - F_X is nondecreasing
  - F_X is right-continuous, i.e. $\lim_{h \downarrow 0} F_X(x+h) = F_X(x)$
  - $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$
- *(Inversion)* Let F be a function satisfying the three properties above. Then there exists some probability space $(\Omega, \mathcal{F}, I\!\!P)$ and a real-valued RV X defined on it such that F is the distribution function of X
  - Key to proof: consider $[0,1]$ with the Borel σ-field and the uniform probability measure
  - Define the “inverse" of F

$$\varphi(u) \doteq \inf\{x : F(x) \geq u\}$$

Change of Variables

- Let X be a RV on $(\Omega, \mathcal{F}, I\!\!P)$ and let $g : I\!\!R \to I\!\!R$ be a Borel-measurable function.
- If g is nonnegative or $I\!\!E|g(X)| < \infty$ then

$$I\!\!E(g(X)) = \int_{I\!\!R} g(x)dI\!\!P_X(x)$$

- Special cases:
    - Discrete RV: $I\!\!E(g(X)) = \sum_i g(x_i)I\!\!P(X = x_i)$
    - Absolutely continuous RV: $I\!\!E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$

Almost Sure Convergence

- Let $X_1, X_2, \ldots$ be a sequence of RV's on $(\Omega, \mathcal{F}, \mathbb{P})$ and let X be another RV on this space.
- **X_n converges to X almost surely** as $n \to \infty$ if there exists $A \in \mathcal{F}$ with $\mathbb{P}(A) = 1$ such that $X_n(\omega) \to X(\omega)$ for all $\omega \in A$.
- If $X_n \to X$ a.s. and f is continuous then $f(X_n) \to f(X)$ a.s.
- *Strong Law of Large Numbers*: if $\{X_n\}$ is an i.i.d. sequence with finite mean μ then

$$\frac{\sum_{i=1}^{n} X_i}{n} \to \mu \text{ a.s. as } n \to \infty$$

Complete Probability Space

- We say that $(\Omega, \mathcal{F}, \mathbb{P})$ is a **complete probability space** if $B \in \mathcal{F}$ with $\mathbb{P}(B) = 0$ implies that $N \in \mathcal{F}$ for any $N \subset B$.
- Any probability space can be completed by adding to $\mathcal{F}$ all subsets of sets of probability 0
- *We will always assume the probability space is complete*
- Completeness guarantees that an a.s. limit of a RV is itself an RV

Convergence in Probability

- Let $X_1, X_2, \ldots$ be a sequence of RV's on $(\Omega, \mathcal{F}, I\!\!P)$ and let X be another RV on this space.
- **X_n converges to X in probability** as $n \to \infty$ if for any $\epsilon > 0$,

$$\lim_{n\to\infty} I\!\!P(|X_n - X| > \epsilon) = 0$$

Convergence in Probability & A.S. Convergence

- If $X_n \to X$ a.s. then $X_n \to X$ in probability
- If $X_n \to X$ in probability then there exists a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \to X$ a.s. as $k \to \infty$

Borel-Cantelli Lemmas

- Let $A_k \in \mathcal{F}, k = 1, 2, \ldots$ and define

$$A^\infty \doteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

- A^∞ is the set of outcomes that occur infinitely often (i.o.)
- **First BC Lemma:**
  If $\sum_{k=1}^{\infty} I\!\!P(A_k) < \infty$ then $I\!\!P(A^\infty) = 0$
- **Second BC lemma:**
  If $\{A_k\}$ are *independent* and $\sum_{k=1}^{\infty} I\!\!P(A_k) = \infty$ then $I\!\!P(A^\infty) = 1$

Example using BC Lemmas

- Let $X_1, X_2, \ldots$ be i.i.d. exponential RV's with rate 1, i.e. $\mathbb{P}(X > x) = e^{-x}$
- Show that

$$\mathbb{P}(X_n > \alpha \log n \text{ for infinitely many } n) = \begin{cases} 0, & \alpha > 1, \\ 1, & \alpha \leq 1 \end{cases}$$

- Define

$$L = \limsup_{n \to \infty} \frac{X_n}{\log n}$$

and show that $L = 1$ a.s.