Last Time

- Probability measure and probability space
- Random variables
- Simple random variables and approximation
- σ-field generated by a random variable

Today's lecture: Section 1.2.3

Definition of Expectation

- Goal: for a RV X on $(\Omega, \mathcal{F}, I\!\!P)$ define

$$I\!\!E(X) = \int_\Omega X(\omega) dI\!\!P(\omega)$$

- Step 1: define for indicator RV's

$$I\!\!E(I_A) = \int_\Omega I_A(\omega) dI\!\!P(\omega) \doteq I\!\!P(A)$$

- Step 2: define for nonnegative simple RV's, $X(\omega) = \sum_{i=1}^n c_i I_{A_i}(\omega)$

$$I\!\!E(X) = \sum_{i=1}^n c_i I\!\!P(A_i)$$

Definition of Expectation (cont.)

- Step 3: define for nonnegative RV's, X
  Let $X_n, n = 1, 2, \ldots$ be a sequence of nonnegative simple RV's such that $X_n(\omega) \to X(\omega)$ as $n \to \infty$ for each $\omega \in \Omega$.
  Then define

$$\mathbb{E}(X) \doteq \lim_{n \to \infty} \mathbb{E}(X_n)$$

- Step 4: define for general RV's, X
  If either $\mathbb{E}(X_+) < \infty$ or $\mathbb{E}(X_-) < \infty$ then

$$\mathbb{E}(X) = \mathbb{E}(X_+) - \mathbb{E}(X_-)$$

  If $\mathbb{E}(X_+) = \mathbb{E}(X_-) = \infty$ then $\mathbb{E}(X)$ is not defined

Integrability

- $\mathbb{E}(X)$ is always defined for a nonnegative RV X, but possible that $\mathbb{E}(X) = +\infty$
- A random variable X is **integrable** if

$$\mathbb{E}|X| = \mathbb{E}(X_+) + \mathbb{E}(X_-) < \infty$$

Expectation of Functions of RV's

- If X is a RV and $g : \mathbb{R} \to \mathbb{R}$ is Borel-measurable then $g(X)$ is a RV, and

$$\mathbb{E}(g(X)) = \int_\Omega g(X(\omega))d\mathbb{P}(\omega)$$

- The **n-th moment** of a RV X is given by

$$\mathbb{E}(X^n) = \int_\Omega X^n(\omega)d\mathbb{P}(\omega)$$

- The **variance** of a RV X is

$$\text{Var}(X) = \mathbb{E}(|X - \mathbb{E}(X)|^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

- Note: above is valid only if the expectations are defined

Properties of Expectation

- If $X = c$ a.s. for some $c \in \mathbb{R}$ then $\mathbb{E}(X) = c$
- *Linearity*: if $X_1, \ldots, X_n$ are integrable RV's on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ then $\sum_{i=1}^n \alpha_i X_i$ is an integrable RV and

$$\mathbb{E}\Big(\sum_{i=1}^n \alpha_i X_i\Big) = \sum_{i=1}^n \alpha_i \mathbb{E}(X_i)$$

- *Monotonicity*: let X and Y be RV's on $(\Omega, \mathcal{F}, \mathbb{P})$
  - If $X \geq Y$ a.s. then $\mathbb{E}(X) \geq \mathbb{E}(Y)$
  - If $X \geq Y$ a.s. and $\mathbb{E}(X) = \mathbb{E}(Y)$ then $X = Y$ a.s.

Example (G& S exercise 3.4.1)

- A coin is tossed n times (tosses are independent). The probability of landing on head is p.
- A run is a sequence of tosses which result in the same outcome. For example, the sequence HHHTHTTH contains 5 runs.
- What is the expected number of runs in n tosses?

Expectation of Discrete RV's

- A RV is **discrete** if it takes values in a countable set $\{x_1, x_2, \ldots\}$, i.e.

$$X(\omega) \in \{x_1, x_2, \ldots\} \text{ for all } \omega \in \Omega$$

- If X is a discrete RV and $\mathbb{E}(X)$ exists then

$$\mathbb{E}(X) = \sum_{i=1}^{\infty} x_i \mathbb{P}(X = x_i)$$

- If X is a discrete RV, g is Borel-measurable, and $\mathbb{E}(g(X))$ exists then

$$\mathbb{E}(g(X)) = \sum_{i=1}^{\infty} g(x_i) \mathbb{P}(X = x_i)$$

Expectation of Absolutely Continuous RV's

- A RV X has a **probability density function** f if

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(u)du, \text{ for all } x \in \mathbb{R}$$

  Such a RV is called **absolutely continuous**.
  Note that f must be nonnegative with $\int_{-\infty}^{\infty} f(u)du = 1$.

- An abs. cont. RV X is integrable if and only if $\int_{-\infty}^{\infty} |x|f(x)dx < \infty$, in which case

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx$$

- If X is an abs. cont. RV and g is Borel-measurable then $g(X)$ is integrable if and only if $\int_{-\infty}^{\infty} |g(x)|f(x)dx < \infty$, in which case $\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$

Jensen's Inequality

- A function $g : \mathbb{R} \to \mathbb{R}$ is **convex** if for all $x, y, \in \mathbb{R}$ and $\lambda \in [0, 1]$

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$$

- If g is convex and X and $g(X)$ are integrable RV's then

$$\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$$

- Examples:

$$\begin{aligned} \mathbb{E}|X| &\geq |\mathbb{E}(X)| \\ \mathbb{E}(X^2) &\geq (\mathbb{E}(X))^2 \end{aligned}$$

Markov's Inequality

- Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, Borel-measurable function with $f(x) > 0$ for all $x > 0$. Then for any RV X and $\epsilon > 0$

$$\mathbb{P}(|X| > \epsilon) \leq \frac{1}{f(\epsilon)} \mathbb{E}(f(|X|))$$

- Special cases:
  - If $X \geq 0$ a.s. then $\mathbb{P}(X \geq \epsilon) \leq \frac{\mathbb{E}(X)}{\epsilon}$
  - $\mathbb{P}(|X - \mathbb{E}(X)| \geq \epsilon) \leq \frac{\text{Var}(X)}{\epsilon^2}$
  - If $\theta > 0$ then $\mathbb{P}(X \geq \epsilon) \leq \frac{\mathbb{E}(e^{\theta X})}{e^{\theta \epsilon}}$

Schwarz Inequality

Suppose Y and Z on RV's on a probability space and $\mathbb{E}(Y^2) < \infty$, $\mathbb{E}(Z^2) < \infty$.
Then

$$|\mathbb{E}(YZ)| \leq \sqrt{\mathbb{E}(Y^2)}\sqrt{\mathbb{E}(Z^2)}$$