Last Time

- Introduction
- Measurable space
- Generated σ-fields
- Borel σ-field

Today's lecture: Sections 1.1–1.2.2

Probability space

- A **probability space** is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F})$ is a measurable space and $\mathbb{P}$ is a probability measure
- A **probability measure** is a set function $\mathbb{P} : \mathcal{F} \to [0, 1]$ which satisfies:
  - $\mathbb{P}(\Omega) = 1$
  - $0 \leq \mathbb{P}(A) \leq 1$ for all $A \in \mathcal{F}$
  - *Countable additivity*: if $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$ are mutually disjoint (i.e. $A_i \cap A_j = \emptyset, i \neq j$) then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

- If $\mathbb{P}(A) = 1$ then we say A occurs **almost surely (a.s.)**

Specifying the Probability Measure $\mathbb{P}$

- Countable Ω
  - Set $\mathcal{F} = 2^{\Omega}$
  - Define p_ω for each $\omega \in \Omega$, such that $0 \le p_\omega \le 1$ and $\sum_{w \in \Omega} p_w = 1$
  - Then $\mathbb{P}(A) \doteq \sum_{\omega \in A} p_w$ defines a probability measure on $(\Omega, 2^{\Omega})$
- Uncountable Ω
  - Consider a set of generators $\{A_\alpha : \alpha \in \Gamma\}$ with $\mathcal{F} = \sigma(\{A_\alpha : \alpha \in \Gamma\})$
  - Define a probability measure $\mathbb{P}(A_\alpha)$ for all $A_\alpha, \alpha \in \Gamma$
  - Then (under mild conditions) $\mathbb{P}$ extends uniquely to a probability measure on $(\Omega, \mathcal{F})$ (see Rosenthal, Section 2.3)

Some properties of $I\!P$

Let $(\Omega, \mathcal{F}, I\!P)$ be a probability space and $A, B, A_i, B_i \in \mathcal{F}$, $i = 1, 2, \ldots$

- $I\!P(A^c) = 1 - I\!P(A)$
- $I\!P(A \cup B) = I\!P(A) + I\!P(B) - I\!P(A \cap B)$
- *Monotonicity*: if $A \subset B$ then $I\!P(A) \leq I\!P(B)$
- *Countable subadditivity*: if $A \subset \cup_{i=1}^{\infty} A_i$ then $I\!P(A) \leq \sum_{i=1}^{\infty} I\!P(A_i)$
- *Continuity from below*: if $A_i \uparrow A$ ($A_1 \subset A_2 \subset \cdots$ and $\cup_{i=1}^{\infty} A_i = A$) then $I\!P(A_i) \uparrow I\!P(A)$
- *Continuity from above*: if $B_i \downarrow B$ ($B_1 \supset B_2 \supset \cdots$ and $\cap_{i=1}^{\infty} B_i = B$) then $I\!P(B_i) \downarrow I\!P(B)$

Definition of Random Variable

- A **random variable** is a real-valued $\mathcal{F}$-measurable function on $(\Omega, \mathcal{F})$
- That is, $X : \Omega \to I\!R$ satisfies

$$X^{-1}(B) \doteq \{\omega : X(\omega) \in B\} \in \mathcal{F}, \text{ for all } B \in \mathcal{B}$$

  Equivalently,

$$X^{-1}((-\infty, \alpha]) \doteq \{\omega : X(\omega) \leq \alpha\} \in \mathcal{F}, \text{ for all } \alpha \in I\!R$$

- Special case: $(\Omega, \mathcal{F}) = (I\!R^n, \mathcal{B}^n)$. A real-valued $\mathcal{B}^n$-measurable function on $(I\!R^n, \mathcal{B}^n)$ is called **Borel-measurable** (or a Borel function)
- Notation: often write $\{\omega : X(\omega) \in B\}$ as $\{X \in B\}$

Definition of Random Vector

- A **random vector** is an $\mathbb{R}^n$-valued $\mathcal{F}$-measurable function on $(\Omega, \mathcal{F})$
- That is, $X = (X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$ satisfies

$$X^{-1}(B) \doteq \{\omega : X(\omega) \in B\} \in \mathcal{F}, \text{ for all } B \in \mathcal{B}^n$$

Equivalently, for all $\alpha_i \in \mathbb{R}, i = 1, \ldots, n$,

$$\{\omega : X_1(\omega) \leq \alpha_1, \ldots, X_n(\omega) \leq \alpha_n\} \in \mathcal{F}$$

- Note: $X = (X_1, \ldots, X_n)$ is a random vector if and only if X_i is a random variable for each $i = 1, \ldots, n$

Simple Functions

- **Indicator function (RV)** of a set:

$$I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

- **Simple function (RV)**:

$$\sum_{i=1}^{n} c_i I_{A_i}(w),$$

where $c_1, \ldots, c_n \in \mathbb{R}$.
Note: can take $\{A_i\}$ to be mutually disjoint

Approximation with Simple Functions

- For any RV X there exists a sequence of simple RV's $X_n, n = 1, 2, \ldots$ such that $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$
- Step 1: define

$$f_n(x) = nI_{\{x>n\}} + \sum_{k=0}^{n2^n-1} k2^{-n} I_{(k2^{-n},(k+1)2^{-n}]}(x)$$

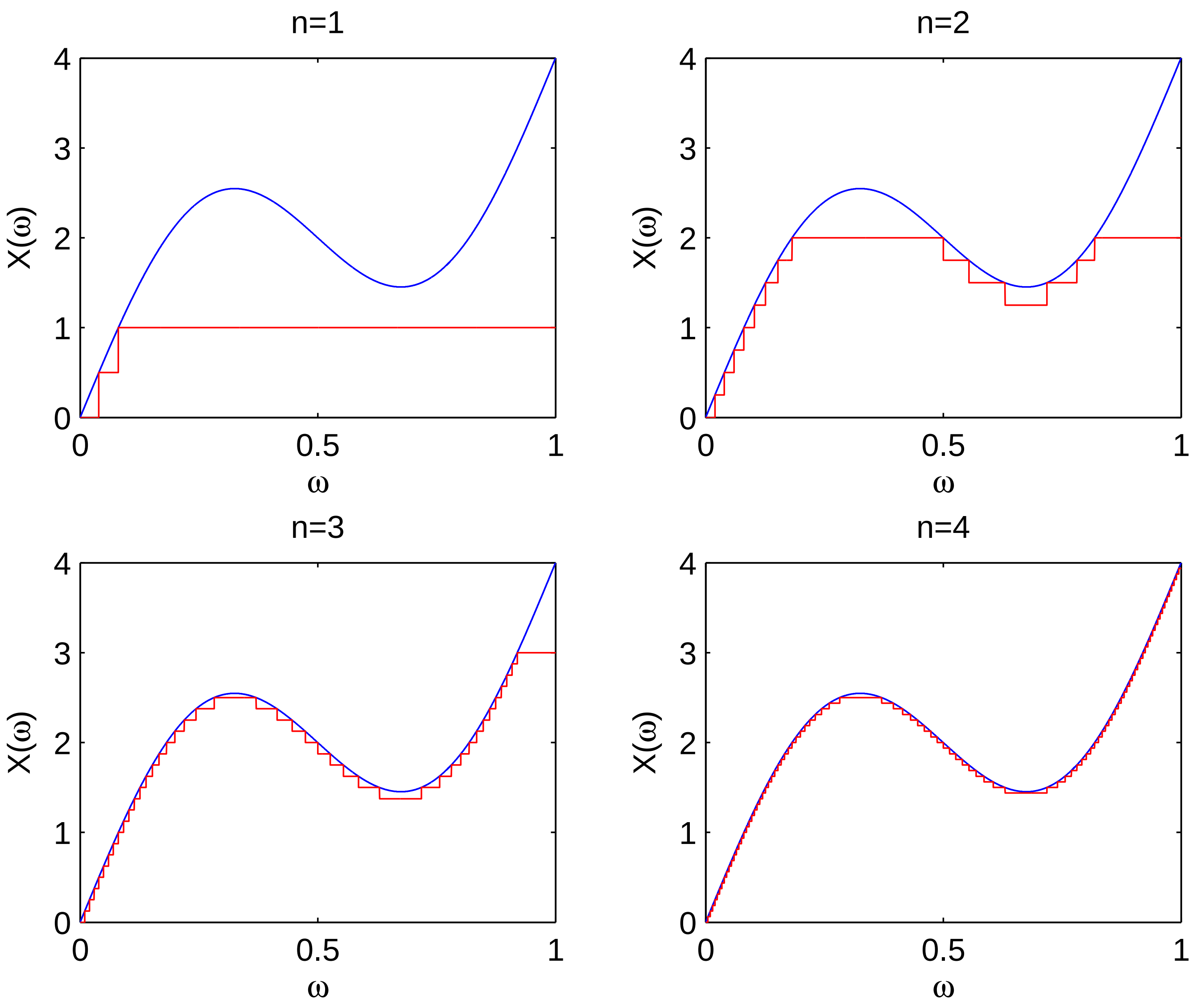
- Step 2: if $X \geq 0$, set

$$X_n(\omega) = f_n(X(\omega))$$

- Step 3: in general, write $X = X_+ - X_-$ and set

$$X_n = f_n(X_+) - f_n(X_-)$$

Approximation with Simple Functions - illustration



Closure Properties of RV's

Let $(\Omega, \mathcal{F})$ be a measurable space and let $X_1, X_2, \ldots$ be a sequence of RV's on it.

- If $\lim_{n\to\infty} X_n(\omega)$ exists and is finite for all $w \in \Omega$ then $\lim_{n\to\infty} X_n$ is a RV
- If $g : \mathbb{R}^n \to \mathbb{R}$ is Borel-measurable, then $g(X_1, \ldots, X_n)$ is a random variable
- Special cases: the following are random variables
  - $|X|$
  - $\sum_{i=1}^n \alpha_i X_i, \alpha_i \in \mathbb{R}$
  - $\prod_{i=1}^n X_i$
  - $\max(X_1, \ldots, X_n)$ and $\min(X_1, \ldots, X_n)$
  - $X_+ \doteq \max(X, 0)$ and $X_- \doteq -\min(X, 0)$

σ-field generated by a RV

- The **σ-field generated by a RV X**, denoted $\sigma(X)$, is the smallest σ-field $\mathcal{G}$ ($\subset \mathcal{F}$) for which X is $\mathcal{G}$-measurable
- Can show that

$$\begin{aligned} \sigma(X) &= \sigma(\{X \leq \alpha\}_{\alpha \in \mathbb{R}}) \\ &= \sigma(\{X \in B\}_{B \in \mathcal{B}}) \end{aligned}$$

- If $X_1, \ldots, X_n$ are random variables on $(\Omega, \mathcal{F})$ then $\sigma(X_i, i = 1, \ldots, n)$ is the smallest σ-field containing $\sigma(X_i)$ for all $i = 1, \ldots, n$

Example: Exercise 1.2.9

- Consider a sequence of two coin tosses, $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^{\Omega}$
- $X_0 = 4$
- $X_1 = 2X_0 I_{\{\omega_1 = H\}} + 0.5 X_0 I_{\{w_1 = T\}}$
- $X_2 = 2X_1 I_{\{\omega_2 = H\}} + 0.5 X_1 I_{\{w_2 = T\}}$
- Find $\sigma(X_0), \sigma(X_1), \sigma(X_2)$

σ-fields as Information

- **$\boldsymbol{\sigma(X)}$ contains the events $\boldsymbol{A}$ for which we can say whether $\boldsymbol{\omega \in A}$ or not, based solely on the value of $\boldsymbol{X(\omega)}$**
- A RV X is $\mathcal{G}$-measurable if and only if the information in $\mathcal{G}$ is sufficient to determine the value of X.
- A RV Y is $\sigma(X_1, \ldots, X_n)$-measurable if and only if $Y = g(X_1, \ldots, X_n)$ for some Borel-measurable function g

Effects of Functions on Information

- If $X_1, \ldots, X_n$ are RV's and g is Borel-measurable, then

$$\sigma(g(X_1, \ldots, X_n)) \subseteq \sigma(X_1, \ldots, X_n)$$

- If $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are RV's defined on $(\Omega, \mathcal{F})$ such that
  - $Y_k = g_k(X_1, \ldots, X_n)$ for each $k = 1, \ldots, m$ and some Borel-measurable functions g_k, and
  - $X_i = h_i(Y_1, \ldots, Y_m)$ for each $i = 1, \ldots, n$ and some Borel-measurable functions h_i,

  then

$$\sigma(X_1, \ldots, X_n) = \sigma(Y_1, \ldots, Y_m)$$