

Last Time

- Counting process
- Poisson process
- Equivalent definitions

Today's lecture: Sections 6.2, 6.3

Markov Property of Poisson Process

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ
- Then $\{N(t), t \geq 0\}$ is a homogenous Markov process with
- State space: $\mathbb{S} = \{0, 1, 2, \dots\}$
- Initial distribution: $\pi(\{0\}) = 1$
- Stationary transition probability function:

$$p_t(n + k|n) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, \quad t \geq 0, n, k = 0, 1, 2, \dots$$

- A Poisson process is a strong Markov process

Compensated Poisson Process

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ
- Let $\{\mathcal{F}_t, t \geq 0\}$ be the canonical filtration of N
- Define $M(t) = N(t) - \lambda t$
- The process $\{M(t), t \geq 0\}$ is called a *compensated Poisson process*
- $\{M(t), \mathcal{F}_t\}$ is a martingale
- $\{M^2(t) - \lambda t, \mathcal{F}_t\}$ is a martingale

Law of Large Numbers for Poisson Process

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda > 0$
- Then as $t \rightarrow \infty$

$$\frac{N(t)}{t} \rightarrow \lambda \text{ almost surely}$$

Poisson Approximation

Suppose that for each n , $Z_k^{(n)}$ are independent, nonnegative integer valued random variables that satisfy:

- $\mathbb{P}(Z_k^{(n)} = 1) = p_k^{(n)}$,
- $\mathbb{P}(Z_k^{(n)} \geq 2) = \epsilon_k^{(n)}$,
- $\sum_{k=1}^n p_k^{(n)} \rightarrow \lambda$ as $n \rightarrow \infty$ for some $\lambda \in (0, \infty)$,
- $\sum_{k=1}^n \epsilon_k^{(n)} \rightarrow 0$ as $n \rightarrow \infty$,
- $\max_{k=1, \dots, n} \{p_k^{(n)}\} \rightarrow 0$ as $n \rightarrow \infty$

Then $S_n = \sum_{k=1}^n Z_k^{(n)}$ converges in distribution as $n \rightarrow \infty$ to a Poisson(λ) random variable

Conditional Distribution of Arrival Times

- Let $\{N(t)\}$ be a Poisson process with arrival times T_1, T_2, \dots
- Conditional on the event $\{N(t) = n\}$, the first n arrival times $\{T_k : k = 1, \dots, n\}$ have the joint probability density:

$$f_{T_1, \dots, T_n | N(t)=n}(t_1, \dots, t_n) = \frac{n!}{t^n}, \text{ for } 0 < t_1 < t_2 < \dots < t_n \leq t$$

- That is, conditional on the event $\{N(t) = n\}$, the first n arrival times $\{T_k : k = 1, \dots, n\}$ have the same distribution as the order statistics of a sample of n i.i.d. $\text{Uniform}([0, t])$ random variables
- Corollary: if $s < t$ and $0 \leq m \leq n$ then

$$\mathbb{P}(N(s) = m | N(t) = n) = \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}$$

Thinning a Poisson Process

- Let $\{N(t)\}$ be a Poisson process with rate λ
- Let Y_k be a sequence of i.i.d random variables taking values in $\{1, 2, \dots\}$ that are independent of $\{N(t)\}$
- For $j = 1, 2, \dots$ let $\{N_j(t)\}$ be the counting process that counts only the events for which $Y_k = j$
- Then $\{N_j(t)\}$ is a Poisson process with rate $\lambda P(Y_k = j)$ and the processes $\{N_j(t)\}$ are independent of each other

Superposition of Poisson Processes

- Let $\{N_j(t)\}$, $j = 1, 2, \dots$ be independent Poisson processes with rates λ_j , respectively
- For each $t \geq 0$ let

$$N(t) = \sum_{j=1}^{\infty} N_j(t)$$

- Then $\{N(t)\}$ is a Poisson process with rate $\lambda = \sum_{j=1}^{\infty} \lambda_j$

Nonhomogeneous Poisson Process

- For each time $u \geq 0$, let $\lambda(u)$ denote the instantaneous arrival rate of some process at that time

- Define

$$\Lambda(t) = \int_0^t \lambda(u) du$$

- A counting process $\{X(t)\}$ is called a *nonhomogeneous Poisson process* if
 - $\{X(t)\}$ has independent increments
 - For all $t > s \geq 0$, $X(t) - X(s)$ has a Poisson distribution with mean $\Lambda(t) - \Lambda(s)$
- If $\{N(t), t \geq 0\}$ is a Poisson process with rate 1, then $\{X(t)\}$ is given by the time-changed process

$$X(t) = N(\Lambda(t))$$

Compound Poisson Process

- Let $\{N(t)\}$ be a Poisson process with rate λ
- Let Y_k be a sequence of i.i.d random variables with finite variance that are independent of $\{N(t)\}$

- For $t \geq 0$ let

$$S(t) = \sum_{i=1}^{N(t)} Y_i$$

- Then $\{S(t)\}$ is called a *compound Poisson process*
- $\{S(t)\}$ has stationary and independent increments and is a homogeneous Markov process

Markov Jump Process

- Let $\{N(t)\}$ be a Poisson process with rate λ and jump times T_1, T_2, \dots
- Let X_1, X_2, \dots be a sequence of random variables with $Z_n = \sum_{k=1}^n X_k$
- Assume that
 - $\{Z_n, n = 0, 1, 2, \dots\}$ is a discrete time Markov chain
 - $\{Z_n\}$ is independent of $\{T_n\}$

- For $t \geq 0$ let

$$X(t) = \sum_{j=1}^{N(t)} X_j$$

- Then $\{X(t)\}$ is called a *jump Markov process*