Last Time

- Continuous time Markov processes
- Transition probability functions
- Functions of Markov processes
- Strong Markov property

Today's lecture: Section 6.2

Exponential Random Variables

- A RV X has an exponential distribution with parameter $\lambda > 0$ ($X \sim$ Exp(λ)) if for all $x \geq 0$,

$$\mathbb{P}(X > x) = e^{-\lambda x}$$

- *Memoryless property*: If X has an exponential distribution then

$$\mathbb{P}(X > x + y | X > y) = \mathbb{P}(X > x) \text{ for all } x, y \geq 0$$

- If a continuous RV has the memoryless property, then the RV has an exponential distribution
- If $X_1, X_2, \ldots$ are i.i.d. Exp(λ) then $T = X_1 + \cdots + X_n$ has a Gamma(n, λ) distribution with density

$$f_T(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0$$

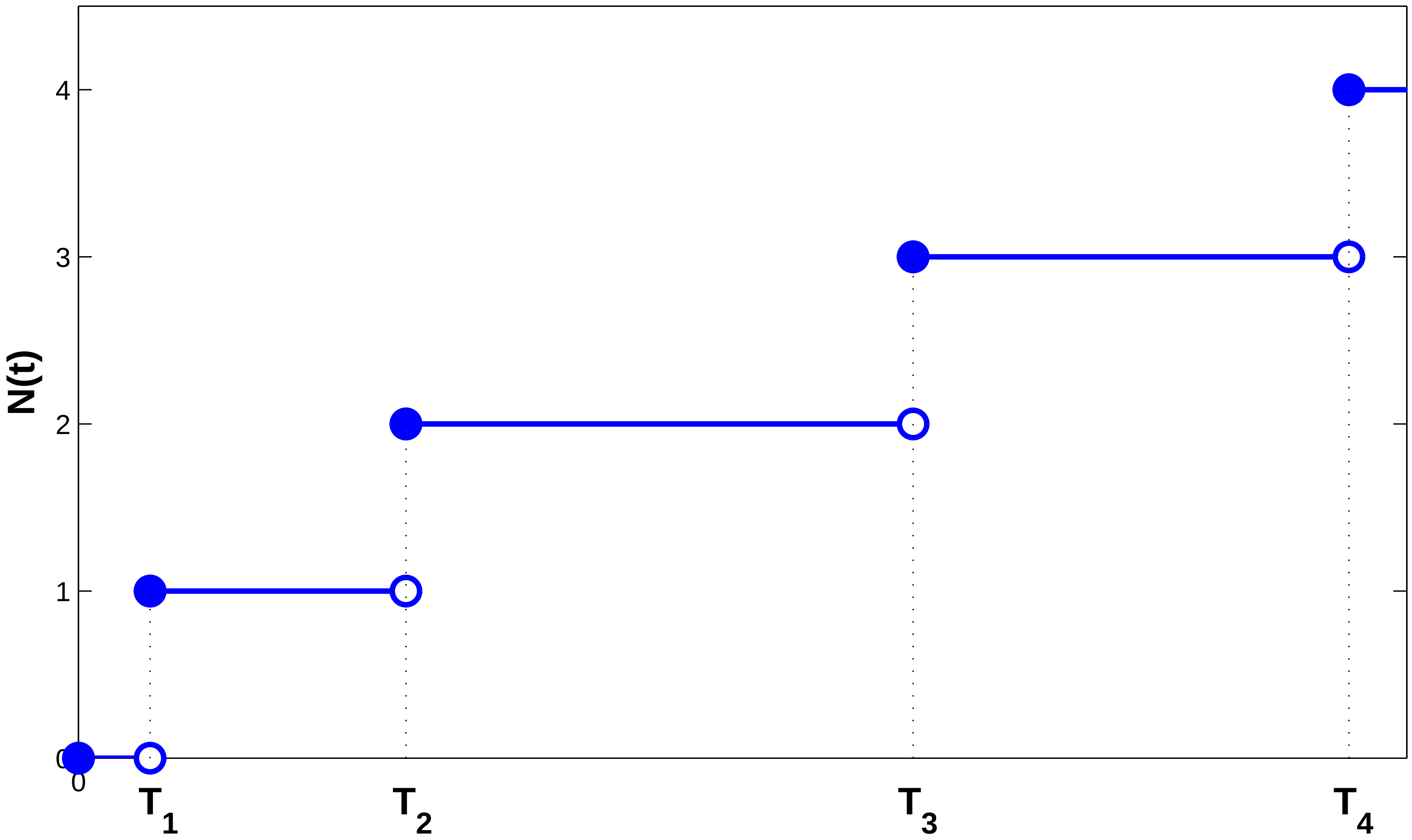
Poisson Process

- Let $X_1, X_2, \ldots$ be i.i.d. Exp(λ) random variables
- Let $T_0 = 0$ and $T_n = X_1 + \cdots + X_n$, $n = 1, 2, \ldots$
- A **Poisson process with rate (or intensity) λ** is the continuous time stochastic process $\{N(t), t \geq 0\}$, where

$$N(t) = \sup\{n \geq 0 : T_n \leq t\} \text{ for } t \geq 0$$

- Interpretation:
  - X_i are the times between occurrences of some event
  - T_n is the time of the nth occurrence of the event
  - $N(t)$ counts the number of occurrences of the event in the time interval $[0, t]$

Illustration: Poisson Process Sample Path



Counting Process

A continuous time stochastic process $\{N(t), t \geq 0\}$ is a *counting process* if

- $N(0) = 0$
- Sample paths of $\{N(t), t \geq 0\}$ are piecewise constant
- Sample paths of $\{N(t), t \geq 0\}$ are nondecreasing
- Sample paths of $\{N(t), t \geq 0\}$ are right-continuous
- All jump discontinuities are of size one and there are infinitely many of them
- A counting process has state space $\mathbb{S} = \{0, 1, 2, \ldots\}$

Counting Process: Jump Times

Associated with each sample path of a counting process $\{N(t), t \geq 0\}$ are the jump times $0 = T_0 < T_1 < T_2 < \cdots$, which satisfy

$$T_n = \inf\{t \geq 0 : N(t) \geq n\}, \quad n = 0, 1, 2, \ldots,$$

or equivalently

$$N(t) = \sup\{n \geq 0 : T_n \leq t\}, \quad t \geq 0$$

In particular, for any $t \geq 0, n = 0, 1, 2, \ldots$

$$N(t) = n \text{ if and only if } T_n \leq t < T_{n+1}$$

Basic Properties of Poisson Processes

Let $\{N(t), t \geq 0\}$ be a Poisson process. Then:

- $\{N(t), t \geq 0\}$ is a counting process
- $N(t) < \infty$ a.s. for all $t \geq 0$ and $N(t) \to \infty$ as $t \to \infty$ a.s.
- For each $t \geq 0$, $N(t)$ has a Poisson distribution with parameter λt, i.e.

$$\mathbb{P}(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

- For each $t > s \geq 0$, $N(t) - N(s)$ has a Poisson distribution with parameter $\lambda(t - s)$

Poisson Process: Equivalent Definition 1

A *counting process* $\{N(t), t \geq 0\}$ is a Poisson process if and only if

- $\{N(t), t \geq 0\}$ has independent increments, and
- $\{N(t), t \geq 0\}$ has stationary increments

Furthermore, there exists $\lambda > 0$ such that for all $t > s \geq 0$, $N(t) - N(s)$ has a Poisson distribution with parameter $\lambda(t - s)$

Poisson Process: Equivalent Definition 2

- Let $\{N(t), t \geq 0\}$ be a counting process and let $X_1, X_2, \ldots$ be the RV's representing the times between jumps
- $\{N(t), t \geq 0\}$ is a Poisson process with rate λ if and only if $X_1, X_2, \ldots$ are i.i.d. Exp(λ) random variables
- In particular, if $T_n = X_1 + \cdots X_n$ then T_n has a Gamma distribution with parameters n and λ

Poisson Process: Equivalent Definition 3

A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate λ if and only if

- $\{N(t), t \geq 0\}$ is continuous in probability
- For any positive integer k, $0 < t_1 < \cdots < t_k$, and nonnegative integers $n_1, n_2, \ldots n_k$, and $h > 0$, both

$$\begin{aligned} I\!\!P(N(t_k+h) - N(t_k) = 1 | N(t_j) = n_j, j \leq k) &= \lambda h + o(h), \text{ and} \\ I\!\!P(N(t_k+h) - N(t_k) \geq 2 | N(t_j) = n_j, j \leq k) &= o(h) \end{aligned}$$

Notation: $o(h)$ denotes a function $g(h)$ which satisfies

$$\frac{g(h)}{h} \rightarrow 0 \text{ as } h \downarrow 0$$

Markov Property of Poisson Process

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ
- Then $\{N(t), t \geq 0\}$ is a homogenous Markov process with
- State space: $\mathbb{S} = \{0, 1, 2, \ldots\}$
- Initial distribution: $\pi(\{0\}) = 1$
- Stationary transition probability function:

$$p_t(n+k|n) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad t \geq 0, n, k = 0, 1, 2, \ldots$$

- A Poisson process is a strong Markov process

Compensated Poisson Process

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ
- Let $\{\mathcal{F}_t\}$ be the canonical filtration of N
- Define $M(t) = N(t) - \lambda t$
- The process $\{M(t), t \geq 0\}$ is called a *compensated Poisson process*
- $\{M(t), \mathcal{F}_t\}$ is a martingale
- $\{M^2(t) - \lambda t, \mathcal{F}_t\}$ is a martingale