Last Time

- Discrete time Markov chains
- Transition probabilities
- Chapman-Kolmogorov equations
- Initial distribution
- Strong Markov property

Today's lecture: Section 6.1

Continuous Time Setup

- Let $\mathbb{S}$ be a closed subset of $I\!R$ and let $\mathcal{B}$ be the corresponding Borel σ-field
- Let $\{X_t, t \geq 0\}$ be a continuous time SP on $(\Omega, \mathcal{F}, I\!P)$ with state space $\mathbb{S}$, i.e. X_t takes values in $\mathbb{S}$ for all t
- Let $\{\mathcal{F}_t\}$ be the canonical filtration of $\{X_t\}$ ($\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$)

Continuous Time Markov Process

- The SP $\{X_t\}$ is a (continuous time) **Markov process** if for all $t, s \geq 0$ and any $A \in \mathcal{B}$

$$I\!\!P(X_{t+s} \in A | \mathcal{F}_t) = I\!\!P(X_{t+s} \in A | X_t) \text{ a.s.}$$

- Equivalent condition: for all $t, s \geq 0$ and any bounded, measurable function $f : \mathbb{S} \mapsto I\!\!R$

$$I\!\!E[f(X_{t+s}) | \mathcal{F}_t] = I\!\!E[f(X_{t+s}) | X_t] \text{ a.s.}$$

Transition Probability Function

A function $p : [0, \infty) \times \mathcal{B} \times \mathbb{S} \mapsto [0, 1]$ is a **(regular) stationary transition probability function** if:

- For any $t \geq 0$ and $x \in \mathbb{S}$, $p_t(\cdot|x)$ is a probability measure on $(\mathbb{S}, \mathcal{B})$
- For any $x \in \mathbb{S}$ and $A \in \mathcal{B}$, $p_0(A|x) = 1$ if $x \in A$ and 0 otherwise
- For all $A \in \mathcal{B}$, $p_{\cdot}(A|\cdot)$ is Borel-measurable (in t and x)
- *Chapman-Kolmogorov Equations*: for all $t, s \geq 0$, $A \in \mathcal{B}$, and $x \in \mathbb{S}$

$$p_{t+s}(A|x) = \int_{\mathbb{S}} p_t(A|y) p_s(dy|x)$$

Homogeneous Markov Process

- Let p be a stationary transition probability function
- A Markov process $\{X_t, t \geq 0\}$ has transition function p if for all $t, s \geq 0$ and $A \in \mathcal{B}$:

$$\begin{aligned} I\!P(X_{t+s} \in A|\mathcal{F}_t) &= p_s(A|X_t) \text{ a.s.; that is,} \\ I\!P(X_{t+s} \in A|\mathcal{F}_t)(\omega) &= p_s(A|X_t(\omega)) \text{ a.s.} \end{aligned}$$

Such a Markov process is **(time) homogeneous**.

Initial Distribution

- Let $\{X_t, t \geq 0\}$ be a homogeneous Markov process
- The **initial distribution** of the process, denoted π, is the distribution of X_0, i.e.

$$\pi(A) = I\!P(X_0 \in A), A \in \mathcal{B}$$

- The distribution of X_t for any $t \geq 0$ is determined by the initial distribution π and the transition probability function $\{p_t(A|x) : t \geq 0, A \in \mathcal{B}, x \in \mathbb{S}\}$
- The initial distribution and the transition probability function determine the FDD's of the homogeneous Markov process

Existence of Markov Process

- Given $(\mathbb{S}, \mathcal{B})$, a probability measure π on $(\mathbb{S}, \mathcal{B})$, and a stationary transition probability function p
- There exists a probability space $(\Omega, \mathcal{F}, I\!P)$ and a SP $\{X_t, t \geq 0\}$ defined on it with state space $\mathbb{S}$ such that
  - $\{X_t\}$ is a homogeneous Markov process
  - $\{X_t\}$ has initial distribution π: $I\!P(X_0 \in A) = \pi(A), A \in \mathcal{B}$
  - $\{X_t\}$ has transition function $p_t(A|x)$:

$$I\!P(X_{t+s} \in A | X_t) = p_s(A|X_t) \text{ for all } t, s \geq 0, A \in \mathcal{B}$$

- If the initial distribution satisfies $\pi(\{x\}) = 1$ for some $x \in \mathbb{S}$, we denote $I\!P$ by $I\!P_x$, i.e. $I\!P_x(X_0 = x) = 1$

Stationary, Independent Increments

- Recall
  - stationary increments:

$$X_{t+s} - X_s \stackrel{d}{=} X_t - X_0 \text{ for all } s, t \geq 0$$

  - independent increments:

$$X_{t+s} - X_s \text{ is independent of } \mathcal{F}_s \text{ for all } s, t \geq 0$$

- If a continuous time SP $\{X_t, t \geq 0\}$ has independent increments then it is a Markov process
- If a continuous time SP $\{X_t, t \geq 0\}$ has independent and stationary increments then it is a homogeneous Markov process

Brownian Motion

- Brownian motion $\{W_t\}$ is a homogeneous Markov process with state space $\mathbb{S} = I\!R$ and transition probability function

$$p_t(A|x) = \int_A \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}, \quad A \in \mathcal{B}, \quad x \in I\!R$$

Functions of Markov Processes

- Suppose $f : [0,\infty) \times \mathbb{S} \mapsto I\!R$ is a nonrandom function such that for each $t \geq 0$ the function $f(t,\cdot) : \mathbb{S} \mapsto I\!R$ is invertible
- Suppose that $g : [0,\infty) \mapsto [0,\infty)$ is a nonrandom invertible and strictly increasing function
- If $\{X_t, t \geq 0\}$ is a Markov process
- Then $\{f(t, X_{g(t)}), t \geq 0\}$ is a Markov process
- If $\{X_t, t \geq 0\}$ is a homogeneous Markov process
- Then $\{f(X_{g(t)}), t \geq 0\}$ is a homogeneous Markov process

Strong Markov Property

- Let $\{X_t, t \geq 0\}$ be a homogeneous Markov process with canonical filtration $\{\mathcal{F}_t\}$
- The MP $\{X_t\}$ has the **strong Markov property** if:

$$\begin{aligned} \mathbb{P}(X_{\tau+s} \in A|\mathcal{F}_\tau) &= \mathbb{P}(X_{\tau+s} \in A|X_\tau) \\ &= \mathbb{P}_{X_\tau}(X_s \in A) \text{ a.s.,} \end{aligned}$$

  for all $s \geq 0$, $A \in \mathcal{B}$, and any $\{\mathcal{F}_t\}$-stopping time with $\tau < \infty$ a.s.

- A continuous time Markov process does not necessarily have the strong Markov property
- However, the above equation does hold for any stopping time that takes at most countably many values