Last Time

- Branching processes
- Probability of extinction

Today's lecture: Section 6.1

Elementary Definition: Countable State MC

- Let $\mathbb{S}$ be a countable set
- Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a SP with **state space** $\mathbb{S}$, i.e. each X_n takes values in $\mathbb{S}$
- The SP $\{X_n\}$ is a **Markov chain** if for all n and all $j, i_0, \ldots, i_n \in \mathbb{S}$:

$$\mathbb{P}(X_{n+1} = j | X_0 = i_0, \ldots, X_n = i_n) = \mathbb{P}(X_{n+1} = j | X_n = i_n)$$

- Equivalent condition 1: for all n, m and $j, i_0, \ldots, i_n \in \mathbb{S}$:

$$\mathbb{P}(X_{n+m} = j | X_0 = i_0, \ldots, X_n = i_n) = \mathbb{P}(X_{n+m} = j | X_n = i_n)$$

- Equivalent condition 2: for all $n_0 < n_1 < \cdots < n_k < n$ and $j, i_0, \ldots, i_k \in \mathbb{S}$:

$$\mathbb{P}(X_{n+1} = j | X_{n_0} = i_0, \ldots, X_{n_k} = i_k) = \mathbb{P}(X_{n+1} = j | X_{n_k} = i_k)$$

Transition Probabilities

- A MC $\{X_n\}$ is **(time) homogeneous** if for all n and all $i, j \in \mathbb{S}$:

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i)$$

- The **transition probability** from state i to state j is given by $p_{i,j} = \mathbb{P}(X_1 = j | X_0 = i)$.
- If $\mathbb{S}$ is a finite set, then $P = [p_{i,j} : i, j \in \mathbb{S}]$ is the transition probability matrix
- Note that $\sum_{j \in \mathbb{S}} p_{i,j} = 1$ for all $i \in \mathbb{S}$

Higher Order Transition Probabilities

- Let $\{X_n\}$ be a homogeneous MC
- The **n-step transition probability** from state i to state j is given by $p_{i,j}^n = \mathbb{P}(X_n = j | X_0 = i)$.
- *Chapman-Kolmogorov equations*: for all m, n and $i, j, \in \mathbb{S}$:

$$p_{i,j}^{m+n} = \sum_{k \in \mathbb{S}} p_{i,k}^m p_{k,j}^n$$

Initial Distribution

- Let $\{X_n\}$ be a homogeneous MC
- The **initial distribution** of the MC, denoted π, is the distribution of X_0, i.e.

$$\pi(i) = \mathbb{P}(X_0 = i), i \in \mathbb{S}$$

- The distribution of X_n for any n is determined by the initial distribution π and the transition probabilities $\{p_{i,j} : i, j \in \mathbb{S}\}$

Strong Markov Property

- Let $\{X_n\}$ be a homogeneous MC with transition probabilities $\{p_{i,j} : i, j \in \mathbb{S}\}$
- Let $\{\mathcal{F}_n\}$ the canonical filtration of $\{X_n\}$ ($\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$)
- Then for any $\{\mathcal{F}_n\}$-stopping time τ,

$$\{X_\tau, X_{\tau+1}, X_{\tau+2}, \ldots\}$$

is a homogeneous Markov chain with transition probabilities $\{p_{i,j} : i, j \in \mathbb{S}\}$

Discrete Time Setup

- Let $\mathbb{S}$ be a closed subset of $I\!R$ and let $\mathcal{B}$ be the corresponding Borel σ-field
- Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a SP on $(\Omega, \mathcal{F}, I\!P)$ with state space $\mathbb{S}$, i.e. X_n takes values in $\mathbb{S}$ for all n
- Let $\{\mathcal{F}_n\}$ be the canonical filtration of $\{X_n\}$ ($\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$)

Markov Chain

- The SP $\{X_n\}$ is a (discrete time) **Markov chain** if for all n and any $A \in \mathcal{B}$

$$\mathbb{P}(X_{n+1} \in A|\mathcal{F}_n) = \mathbb{P}(X_{n+1} \in A|X_n) \text{ a.s.}$$

- Equivalent condition 1: for all n, m and any $A \in \mathcal{B}$

$$\mathbb{P}(X_{n+m} \in A|\mathcal{F}_n) = \mathbb{P}(X_{n+m} \in A|X_n) \text{ a.s.}$$

- Equivalent condition 2: for all n and any bounded, measurable function $f : \mathbb{S} \mapsto \mathbb{R}$

$$\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] = \mathbb{E}[f(X_{n+1})|X_n] \text{ a.s.}$$

Homogeneous MC and Transition Probabilities

- A MC $\{X_n\}$ is **(time) homogeneous** if it has a modification which satisfies: for any $A \in \mathcal{B}$:

$$\mathbb{P}(X_{n+1} \in A | X_n) \text{ does not depend on } n$$

- A homogeneous MC $\{X_n\}$ has a **stationary transition probability function**, denoted $p(A|x) : \mathcal{B} \times \mathbb{S} \mapsto [0, 1]$, which satisfies
  - For any $x \in \mathbb{S}$, $p(\cdot|x)$ is a probability measure on $(\mathbb{S}, \mathcal{B})$
  - For any $A \in \mathcal{B}$, $p(A|\cdot)$ is a $\mathcal{B}$-measurable map
  - For any $A \in \mathcal{B}$ and all n,

$$\begin{aligned} p(A|X_n) &= \mathbb{P}(X_{n+1} \in A | X_n) \text{ a.s.; that is,} \\ p(A|X_n(\omega)) &= \mathbb{P}(X_{n+1} \in A | X_n)(\omega) \text{ a.s.} \end{aligned}$$

Initial Distribution

- Let $\{X_n\}$ be a homogeneous MC
- The **initial distribution** of a MC, denoted π, is the distribution of X_0
- That is, π is a probability measure on $(\mathbb{S}, \mathcal{B})$ given for $A \in \mathcal{B}$ by $\pi(A) = \mathbb{P}(X_0 \in A)$.
- The distribution of X_n for any n is determined by the initial distribution π and the transition probability function $\{p(A|x), A \in \mathcal{B}, x \in \mathbb{S}\}$.
- The initial distribution and the transition probability function determine the FDD's of the MC

Existence of MC

- Given $(\mathbb{S}, \mathcal{B})$, a probability measure π on $(\mathbb{S}, \mathcal{B})$, and a function $p(A|x) : \mathcal{B} \times \mathbb{S} \mapsto [0, 1]$ which satisfies:
  - For any $x \in \mathbb{S}$, $p(\cdot|x)$ is a probability measure on $(\mathbb{S}, \mathcal{B})$
  - For any $A \in \mathcal{B}$, $p(A|\cdot)$ is a $\mathcal{B}$-measurable map
- There exists a probability space $(\Omega, \mathcal{F}, I\!P)$ and a SP $\{X_n\}$ defined on it with state space $\mathbb{S}$ such that
  - $\{X_n\}$ is a homogeneous Markov chain
  - $\{X_n\}$ has initial distribution π: $I\!P(X_0 \in A) = \pi(A), A \in \mathcal{B}$
  - $\{X_n\}$ has transition function $p(A|x)$:

$$I\!P(X_{n+1} \in A|X_n) = p(A|X_n) \text{ for all } n, A \in \mathcal{B}$$

- If the initial distribution satisfies $\pi(\{x\}) = 1$ for some $x \in \mathbb{S}$, we denote $I\!P$ by $I\!P_x$, i.e. $I\!P_x(X_0 = x) = 1$

Strong Markov Property

- Let $\{X_n\}$ be a homogeneous MC with canonical filtration $\{\mathcal{F}_n\}$
- The MC $\{X_n\}$ has the **strong Markov property**:

$$\begin{aligned} \mathbb{P}(X_{\tau+m} \in A|\mathcal{F}_\tau) &= \mathbb{P}(X_{\tau+m} \in A|X_\tau) \\ &= \mathbb{P}_{X_\tau}(X_m \in A) \text{ a.s.,} \end{aligned}$$

  for all m, $A \in \mathcal{B}$, and any $\{\mathcal{F}_n\}$-stopping time with $\tau < \infty$ a.s.
- Any discrete time, homogeneous Markov chain satisfies the strong Markov property