Last Time

- Doob decomposition (discrete time)
- Doob-Meyer decomposition (continuous time)
- Quadratic variation of stochastic process
- Quadratic variation of Brownian motion
- Martingale characterization of Brownian motion

Today's lecture: Section 4.6

Branching Processes

- Let N and $N_j^{(n)}$ be i.i.d. non-negative integer valued RV's with finite mean $m = \mathbb{E}(N) < \infty$.
- A **Branching Process** is a discrete time SP $\{Z_n\}$ taking nonnegative integer values such that $Z_0 = 1$ and for any $n = 1, 2, \ldots$

$$Z_n = \sum_{j=1}^{Z_{n-1}} N_j^{(n)},$$

- We take $Z_{n+1} = 0$ if $Z_n = 0$

Interpretation

- Z_n represents the size of the n-th generation of some population
- $N_j^{(n)}$ represents the number of offspring of the j-th individual of the $(n-1)$-st generation
- $m = \mathbb{E}(N)$ is mean number of offspring for each individual

Martingale Properties

- Let $\mathcal{F}_n = \sigma(N_j^{(k)}, k \leq n, j = 0, 1, 2, \ldots)$
- Then $\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = mZ_n$. In particular,
  - $\mathbb{E}(Z_n) = m^n$
  - If $m = 1$, $\{(Z_n, \mathcal{F}_n)\}$ is a martingale
- $\{(m^{-n}Z_n, \mathcal{F}_n)\}$ is a martingale

Probability of Extinction

- Let p_{ex} denote the *probability of extinction*:

$$p_{\text{ex}} = \mathbb{P}(Z_n = 0 \text{ for some } n \geq 0)$$

- Sub-critical process dies off:

$$\text{if } m < 1 \text{ then } p_{\text{ex}} = 1$$

- Critical process dies off:

$$\text{if } m = 1 \text{ and } \mathbb{P}(N = 1) < 1 \text{ then } p_{\text{ex}} = 1$$

- Super-critical process can survive forever:

$$\text{if } m > 1 \text{ then } p_{\text{ex}} < 1$$

Probability of Extinction: Sub-critical Case

- Suppose $m < 1$ and let $X_n = m^{-n} Z_n$
- Since $\{X_n\}$ is a nonnegative martingale
- By the martingale convergence theorem, there exists a random variable X_∞ such that

$$X_n \to X_\infty \text{ a.s. as } n \to \infty$$

- Since $X_\infty < \infty$ a.s. and $m < 1$, $Z_n \to 0$ a.s.
- Since Z_n takes integer values, must have $Z_n = 0$ for some n, so $p_{\text{ex}} = 1$

Probability of Extinction: Critical Case

- Suppose $m = 1$ and $\mathbb{P}(N = 1) < 1$
- Then $\{Z_n\}$ is a martingale and as before there exists a random variable Z_∞ such that

$$Z_n \to Z_\infty \text{ a.s. as } n \to \infty$$

- Proof of Proposition 4.6.5 shows that $Z_\infty = 0$ a.s. and $p_{\text{ex}} = 1$
- Note that since $\mathbb{E}(Z_n) = 1$ for all n, Z_n does not converge in L^1.
- Thus, $\{Z_n\}$ is not uniformly integrable