Last Time

- Martingale inequalities
- Martingale convergence theorem
- Uniformly integrable martingales

Today's lecture: Sections 4.4.1, 5.3

Doob’s Decomposition in Discrete Time

- Let $\{X_n\}$ be a discrete time SP with $\mathbb{E}|X_n| < \infty$ that is adapted to some filtration $\{\mathcal{F}_n\}$
- Then there exists a unique decomposition $X_n = M_n + A_n$ such that
    - $\{M_n, \mathcal{F}_n\}$ is a martingale
    - $\{A_n\}$ is a previsible SP; i.e. A_{n+1} is $\mathcal{F}_n$-measurable
    - $A_0 = 0$
- Proof: define $M_n = X_n - A_n$ where A_n is defined via the recursive equation

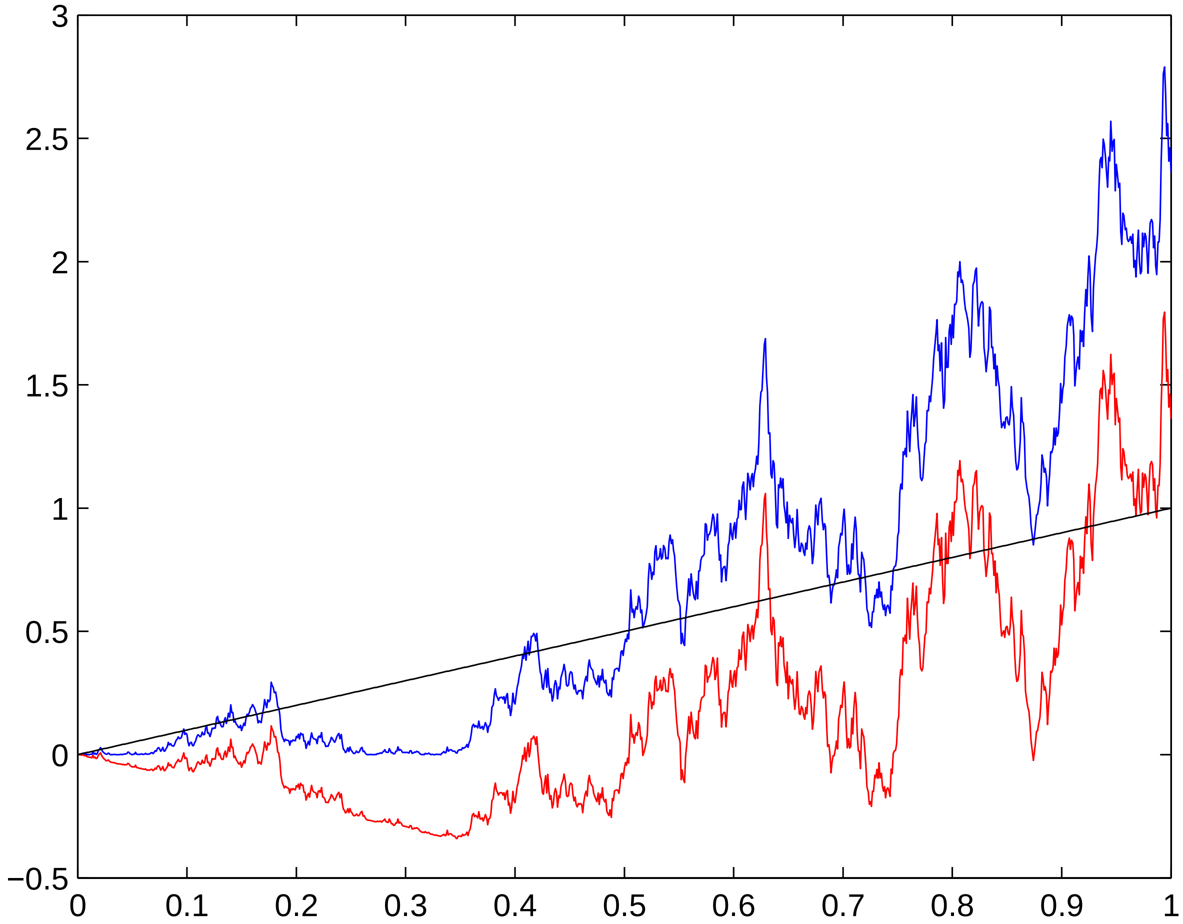
$$A_{n+1} = A_n + \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n)$$

- If $\{X_n\}$ is a submartingale, then $\{A_n\}$ is a nondecreasing process, i.e. $A_{n+1} \geq A_n$ a.s. for all n

Doob-Meyer Decomposition

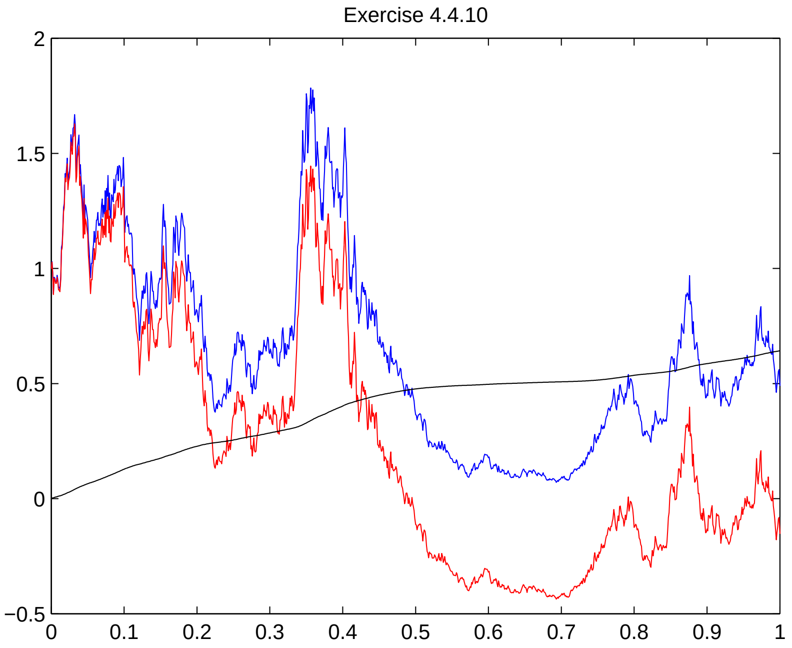
- Let $\{M_t, \mathcal{F}_t\}$ be a *continuous, square integrable martingale* (i.e. $\mathbb{E}(M_t^2) < \infty$ for all $t \geq 0$ and $\{M_t\}$ has continuous paths a.s.)
- Then there exists a unique SP $\{A_t\}$ such that
  - $A_0 = 0$
  - $\{A_t\}$ is adapted to $\{\mathcal{F}_t\}$
  - $\{A_t\}$ has continuous sample paths a.s.
  - $\{A_t\}$ is nondecreasing ($A_t \geq A_s$ a.s. for all $t \geq s \geq 0$)
  - $\{(M_t^2 - A_t, \mathcal{F}_t)\}$ is a martingale
- $\{A_t\}$ is called the *increasing part* associated with $\{M_t\}$, and $\{A_t\}$ is equal to the quadratic variation process of $\{M_t\}$

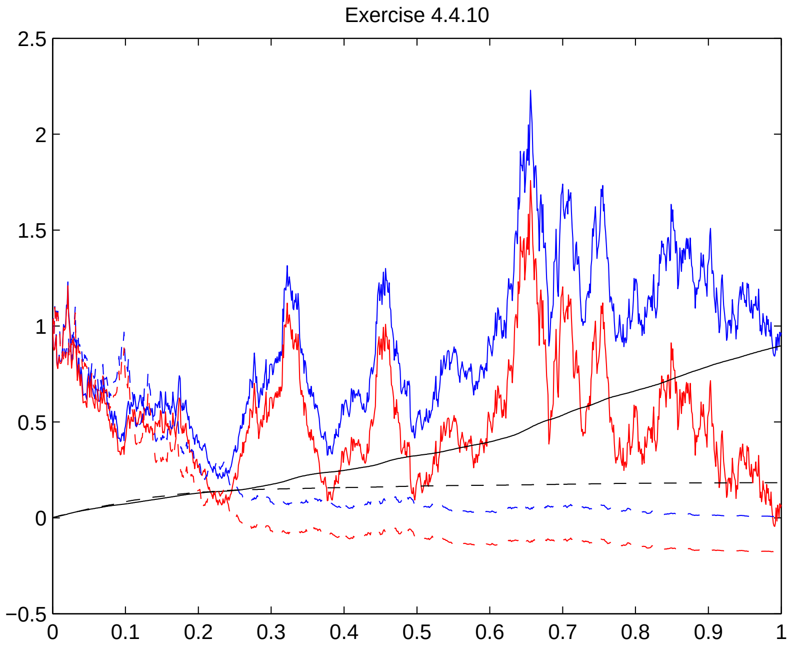
Illustration: Doob-Meyer decomposition of W_t^2



blue: W_t^2, red: $W_t^2 - t$

Illustration: D-M decomposition in Exercise 4.4.10





$$M_t = \exp(W_t - t/2)$$

blue: M_t^2, red: $M_t^2 - A_t$

Variation of a Function

- For $t > 0$, let π be a partition of $[0, t]$:

$$\pi = \{0 = t_0^{(\pi)} < t_1^{(\pi)} < \cdots < t_k^{(\pi)} = t\}$$

and define $||\pi|| = \max_{1 \le i \le k}(t_i^{(\pi)} - t_{i-1}^{(\pi)})$

- For a function $f : [0, t] \mapsto I\!R$ and $p \geq 1$, the **p-th variation of f on $[0, t]$** is defined as

$$V^{(p)}(f) = \lim_{||\pi|| \to 0} \sum_{i=1}^{k} |f(t_i^{(\pi)}) - f(t_{i-1}^{(\pi)})|^p$$

provided the limit exists

- Special cases: $p = 1$ gives *total variation* and $p = 2$ gives *quadratic variation* of f on $[0, t]$

Variation of a Stochastic Process

- The **total variation process** of a SP $\{X_t\}$ is the stochastic process $\{V_t^{(1)}, t \geq 0\}$, where the value at time $t \geq 0$, $V_t^{(1)}(\omega)$, is the total variation of the function $X_s(\omega)$ on the interval $[0, t]$
- The **quadratic variation process** of a SP $\{X_t\}$ is the stochastic process $\{V_t^{(2)}, t \geq 0\}$, where the value at time $t \geq 0$, $V_t^{(2)}(\omega)$, is the quadratic variation of the function $X_s(\omega)$ on the interval $[0, t]$
- (Above definitions apply only in cases where the limits are defined in some sense)

Variation of Continuous, L^2 Martingales

- Let $\{(X_t, \mathcal{F}_t)\}$ be a continuous, square integrable martingale
- The quadratic variation process of $\{X_t\}$, often denoted $\langle X \rangle_t$ or $\langle X, X \rangle_t$, exists and is equal to $\{A_t\}$, the increasing part of the Doob-Meyer decomposition
- That is, $\{(X_t^2 - \langle X \rangle_t, \mathcal{F}_t)\}$ is a martingale
- Also, the total variation of $\{X_t\}$ on any interval is infinite with probability 1

Variation of Brownian Motion

- Let $\{W_t\}$ be a Brownian Motion
- Then for all $t > 0$,

$$\sum_{i=1}^{k} |W(t_i^{(\pi)}) - W(t_{i-1}^{(\pi)})|^2 \to t \text{ in } L^2 \text{ as } ||\pi|| \to 0$$

- That is, the quadratic variation process of Brownian motion is given by $\langle W \rangle_t = t$ a.s.
- Brownian motion accumulates quadratic variation at rate one per unit time
- Informally, $dW_t dW_t = dt$, and also $dW_t dt = 0$ and $dtdt = 0$
- The total variation of Brownian motion on any interval is infinite with probability 1

Definition of $\{\mathcal{F}_t\}$-Brownian Motion

An $\{\mathcal{F}_t\}$-adapted stochastic process $\{W_t, t \geq 0\}$ is a **$\{\mathcal{F}_t\}$-Brownian motion** if:

- $W_0 = 0$ a.s.
- *Independent increments*: for all $0 \leq s \leq t$

$$W_t - W_s \text{ is independent of } \mathcal{F}_s$$

- *Stationary increments*: for all $0 \leq s \leq t$

$$W_t - W_s \text{ has a } N(0, t-s) \text{ distribution}$$

- For almost every ω, the sample path $t \mapsto W_t(\omega)$ is continuous

Martingale Characterization of Brownian Motion

- Suppose $\{(X_t, \mathcal{F}_t)\}$ is a martingale with *continuous* paths
- If $\{(X_t^2 - t, \mathcal{F}_t)\}$ is a martingale
- Then $\{X_t\}$ is a $\{\mathcal{F}_t\}$-Brownian motion