

Last Time

- Reflection principle for BM
- Brownian hitting times
- Running maximum of BM
- Law of large numbers for BM

Today's lecture: Section 4.4.2, 4.5

Doob's Submartingale Inequalities

- If $\{X_t\}$ is a RCLL submartingale then for all $x > 0$ and $t > 0$

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} X_s > x\right) \leq \frac{\mathbb{E}|X_t|}{x}$$

- If $\{X_t\}$ is a martingale then for all $x > 0$ and $t > 0$

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} |X_s| > x\right) \leq \frac{\mathbb{E}|X_t|}{x}$$

- Analogous versions exist for discrete time

Doob's Maximal Inequality

- If either
 - $\{X_t\}$ is a RCLL martingale, or
 - $\{X_t\}$ is a nonnegative RCLL submartingale
- Then for all $p > 1$ and $t \geq 0$

$$\mathbb{E} \left[\left(\sup_{0 \leq s \leq t} |X_s| \right)^p \right] \leq \left(\frac{p}{p-1} \right)^p \mathbb{E} |X_t|^p$$

- Analogous version exists for discrete time

Martingale Convergence Theorem

- If $\{X_t, \mathcal{F}_t\}$ is a RCLL submartingale with

$$\sup_{t \geq 0} \mathbf{E}(X_t^+) < \infty$$

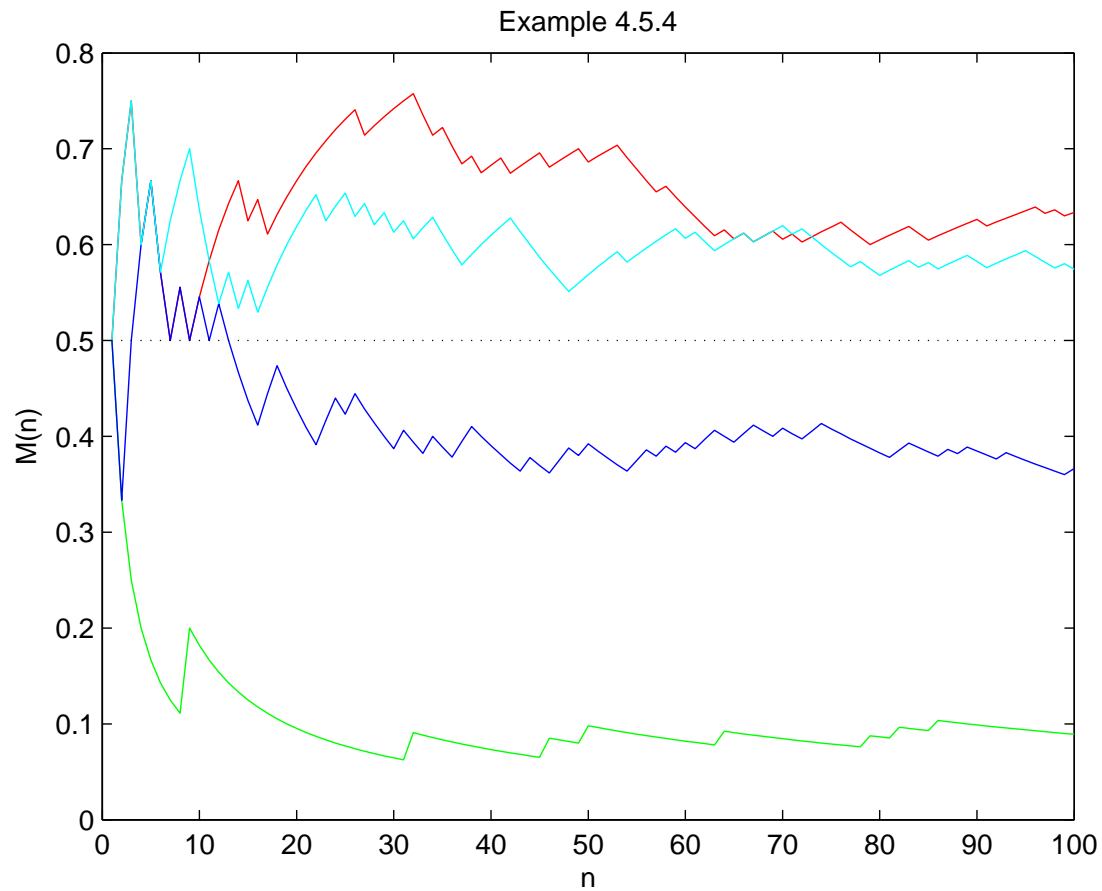
- Then there exists a RV X_∞ such that $\mathbf{E}|X_\infty| < \infty$ and

$$X_t \rightarrow X_\infty \text{ almost surely as } t \rightarrow \infty$$

- In addition,

$$\mathbf{E}|X_\infty| \leq \lim_{t \rightarrow \infty} \mathbf{E}|X_t| < \infty$$

Illustration: Martingale Convergence



See Exercise 4.5.4

Martingale Convergence and Uniform Integrability

- If $\{X_t, \mathcal{F}_t\}$ is a *uniformly integrable* RCLL submartingale with

$$\sup_{t \geq 0} \mathbb{E}(X_t^+) < \infty$$

- Then there exists a RV X_∞ such that $\mathbb{E}|X_\infty| < \infty$ and

$$X_t \rightarrow X_\infty \text{ almost surely as } t \rightarrow \infty$$

- In addition,

$$X_t \rightarrow X_\infty \text{ in } L^1 \text{ as } t \rightarrow \infty$$

- In this case, X_∞ is a *last element*, i.e.

$$\mathbb{E}(X_\infty | \mathcal{F}_t) = X_t \text{ for all } t \geq 0$$

Uniformly Integrable Martingales

If $\{X_t, \mathcal{F}_t\}$ is martingale then the following are equivalent:

- $\{X_t\}$ is uniformly integrable
- $\{X_t\}$ has a last element: i.e. there exists an \mathcal{F}_∞ -measurable RV X_∞ satisfying:
 - $\mathbb{E}|X_\infty| < \infty$, and
 - $\mathbb{E}(X_\infty | \mathcal{F}_t) = X_t$ for all $t \geq 0$
- X_t converges almost surely as $t \rightarrow \infty$
- X_t converges in L^1 as $t \rightarrow \infty$

Levy Martingales

- Let Y be a RV with $\mathbb{E}|Y| < \infty$ and let $\{\mathcal{F}_t\}$ be some filtration. Define $\mathcal{F}_\infty = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$
- Then $\{\mathbb{E}(Y|\mathcal{F}_t), t \geq 0\}$ is a uniformly integrable martingale and as $t \rightarrow \infty$,

$$\mathbb{E}(Y|\mathcal{F}_t) \rightarrow \mathbb{E}(Y|\mathcal{F}_\infty),$$

almost surely and in L^1

- All uniformly integrable martingales are of this form