Math136 / Stat219 Course Goals

- Basic concepts and definitions of measure-theoretic probability and stochastic processes
- Properties of key stochastic processes and their applications, especially Brownian motion
- Key results and common techniques of proof
- Preparation for further study (especially for Math 236: stochastic differential equations)

Today's lecture: Sections 1.1

Why Measure-Theoretic Probability?

- Mathematical models of physical processes
- Outcome is uncertain or “random”
- Probability = “Language”
- Measure Theory = “Grammar”
- Measure theory allows us to consider
  - General random variables
  - Arbitrary probability spaces

Measurable space

- ω: **outcome** of random experiment
- Ω: **sample space** - set of all possible outcomes
- A collection, $\mathcal{F}$, of subsets of Ω is a **σ-field** (aka σ-algebra) if:
  - $\Omega \in \mathcal{F}$
  - if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
  - if $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
- A **measurable space** is a pair $(\Omega, \mathcal{F})$ is a σ-field of subsets of Ω
- $\mathcal{F}$: **event space** - all events of interest; all events we will assign probabilities to

Choosing the σ-field $\mathcal{F}$

- Countable sample space

$$\mathcal{F} = 2^{\Omega}$$

- Uncountable sample space
  - 2^{Ω} is too big (for an example, see Rosenthal Section 1.2)
  - Require inclusion of certain “nice" sets
  - Take $\mathcal{F}$ to be smallest σ-field that includes these sets

Generated σ-field

- Let Γ be an arbitrary index set, and let $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$ be a collection of subsets of Ω
- The **σ-field generated by $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$**, denoted $\sigma(\{\mathcal{A}_\alpha : \alpha \in \Gamma\})$, is the smallest σ-field that contains the collection $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$
- $\sigma(\{\mathcal{A}_\alpha : \alpha \in \Gamma\})$ is the intersection of all σ-fields that contain $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$
- Note: intersection of an arbitrary collection of σ-fields is a σ-field
  - But an arbitrary union of σ-fields is not necessarily a σ-field

Borel σ-field

- The **Borel σ-field on $\mathbb{R}$** is $\mathcal{B} \doteq \sigma(\{(a,b) : a < b \in \mathbb{R}\})$
- Some equivalent definitions:

$$\begin{aligned}
\mathcal{B} &= \sigma(\{[a,b] : a < b \in \mathbb{R}\}) = \sigma(\{(a,b] : a < b \in \mathbb{R}\}) \\
&= \sigma(\{(a,b) : a < b \in \mathbb{Q}\}) = \sigma(\{[a,b] : a < b \in \mathbb{Q}\}) \\
&= \sigma(\{(-\infty,b] : b \in \mathbb{R}\}) = \sigma(\{(-\infty,b] : b \in \mathbb{Q}\}) \\
&= \sigma(\{\text{open sets of } \mathbb{R}\})
\end{aligned}$$

- The **Borel σ-field on $\mathbb{R}^n$** is $\mathcal{B}^n \doteq \sigma(\{(a_1,b_1) \times \cdots \times (a_n,b_n) : a_i, b_i \in \mathbb{R}, i = 1, \ldots, n\})$
- In general, the Borel σ-field on a space Ω is defined as $\sigma(\{\text{open sets of } \Omega\})$

Example: showing two σ-fields are equal

- Consider the σ-fields
  - $\mathcal{B} = \sigma(\{(a, b) : a < b \in I\!R\})$
  - $\mathcal{G} = \sigma(\{(a, b], a < b \in I\!R\})$
- $(a, b) \in \mathcal{G}$ for all $a < b \in I\!R \Rightarrow \mathcal{B} \subseteq \mathcal{G}$
  - $\{b\} = \cap_{n=1}^{\infty}(b - \frac{1}{n}, b] \in \mathcal{G}$
  - $(a, b) = (a, b] \setminus \{b\} \in \mathcal{G}$
- $(a, b] \in \mathcal{B}$ for all $a < b \in I\!R \Rightarrow \mathcal{G} \subseteq \mathcal{B}$
  - $\{b\} = \cap_{n=1}^{\infty}(b - \frac{1}{n}, b + \frac{1}{n}) \in \mathcal{B}$
  - $(a, b] = (a, b) \cup \{b\} \in \mathcal{B}$