Last Time

- Continuous time stopping times
- Optional stopping theorem
- Stopped σ-field
- Optional sampling theorem

Today's lecture: Section 5.2

Regeneration Properties of Brownian Motion

- Recall: if $\{W_t\}$ is a Brownian motion and T is any constant $(0 \leq T < \infty)$ then

$$\{W_{T+t} - W_T, t \geq 0\}$$

  is a Brownian motion

- If $\{W_t\}$ is a Brownian motion and τ is a stopping time with respect to its canonical filtration $\{\mathcal{F}_t\}$ then

$$\{W_{\tau+t} - W_\tau, t \geq 0\}$$

  is a Brownian motion, and it is independent of the stopped σ-field $\mathcal{F}_\tau$

Reflection Principle for Brownian Motion

- Let $\{W_t\}$ be a BM and $\{\mathcal{F}_t\}$ its canonical filtration
- For fixed $\alpha > 0$ define

$$\tau_\alpha = \inf\{t \geq 0 : W_t = \alpha\}$$

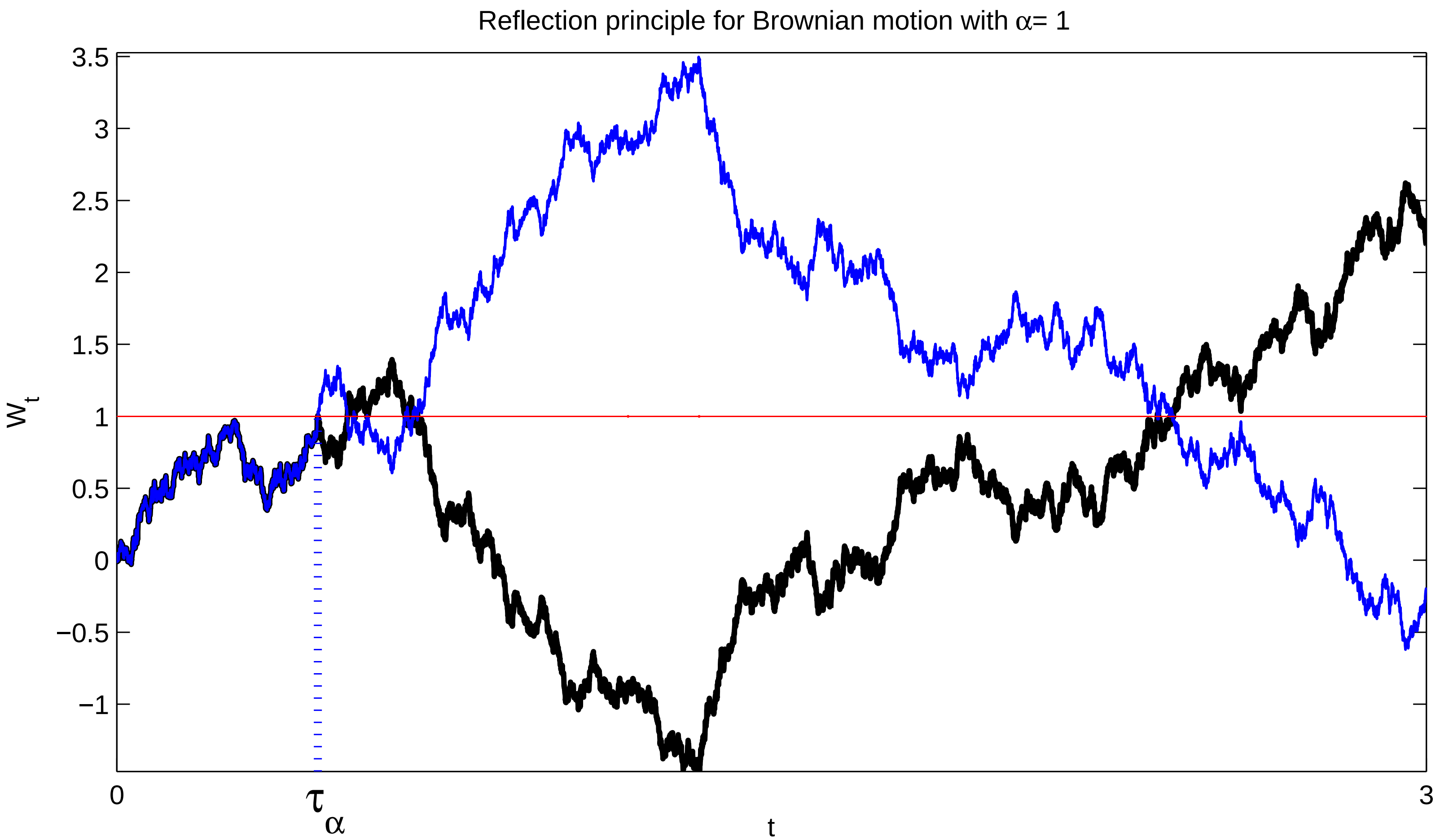
- Define a process $\{\tilde{W}_t, t \geq 0\}$ by

$$\tilde{W}_t = \begin{cases} W_t & t \leq \tau_\alpha, \\ 2\alpha - W_t & t > \tau_\alpha \end{cases}$$

- Then $\{\tilde{W}_t, t \geq 0\}$ is a Browian motion
- The reflection principle implies for $w \leq \alpha$

$$I\!\!P(\tau_\alpha \leq t, W_t \leq w) = I\!\!P(W_t \geq 2\alpha - w)$$

Illustration: Reflection Principle



W_t **and** $\tilde{W}_t$ **for** $\alpha = 1$

Properties of τ_α

- Let $\{W_t\}$ be a Brownian motion and $\{\mathcal{F}_t\}$ its canonical filtration
- For fixed $\alpha > 0$ define $\tau_\alpha = \inf\{t \geq 0 : W_t = \alpha\}$
- The distribution function of τ_α is

$$\mathbb{P}(\tau_\alpha \leq t) = \frac{2}{\sqrt{2\pi}} \int_{\frac{\alpha}{\sqrt{t}}}^{\infty} e^{-u^2/2} du,$$

  with density

$$f_{\tau_\alpha}(t) = \frac{\alpha}{\sqrt{2\pi} t^{3/2}} e^{-\frac{\alpha^2}{2t}}, t \geq 0$$

- In particular, $\mathbb{P}(\tau_\alpha < \infty) = 1$ and $\mathbb{E}(\tau_\alpha) = \infty$

Running Maximum of Brownian Motion

- Let $\{W_t\}$ be a Brownian motion and for fixed $\alpha > 0$ define

$$\tau_\alpha = \inf\{t \geq 0 : W_t = \alpha\}$$

- Since

$$\left\{\max_{0\leq s\leq t} W_s \geq \alpha\right\} = \{\tau_\alpha \leq t\},$$

we have

$$\mathbb{P}\left(\max_{0\leq s\leq t} W_s \geq \alpha\right) = \mathbb{P}(\tau_\alpha \leq t),$$

which can be computed using distribution of τ_α

Limiting Properties of Brownian Motion Paths

- Let $\{W_t\}$ be a Brownian motion
- For $\alpha > 0$ define $\tau_\alpha = \inf\{t \geq 0 : W_t = \alpha\}$ and recall that $I\!P(\tau_\alpha < \infty) = 1$
- It follows that

$$\limsup_{t\to\infty} W_t = +\infty \text{ a.s.},$$
$$\liminf_{t\to\infty} W_t = -\infty \text{ a.s.}$$

Law of the Iterated Logarithm

- Let $\{W_t\}$ be a Brownian motion
- Behavior for large t: almost surely

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log(\log t)}} = 1, \text{ and } \liminf_{t\to\infty} \frac{W_t}{\sqrt{2t\log(\log t)}} = -1$$

- Behavior for t near 0: almost surely

$$\limsup_{t\to 0} \frac{W_t}{\sqrt{2t\log(\log(1/t))}} = 1, \text{ and } \liminf_{t\to 0} \frac{W_t}{\sqrt{2t\log(\log(1/t))}} = -1$$

- **Law of large numbers for Browian motion**:

$$\frac{W_t}{t} \to 0 \text{ a.s. as } t \to \infty$$