Last Time

- Continuous time martingales
- Right-continuous filtrations

Today's lecture: Sections 4.3.1

A Motivating Problem: Gambler's Ruin

- A gambler enters a casino with a dollars
- He keeps playing a game until either he loses all his money or he wins b dollars
- How long does the gambler play?
- How much money does he have when he stops playing?

Discrete Time Stopping Time

- A random variable τ taking values in $\{0, 1, 2, \ldots\} \cup \{+\infty\}$ is a **stopping time** with respect to filtration $\{\mathcal{F}_n\}$ if

$$\{\tau \leq n\} = \{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n \quad \text{for all } n = 0, 1, 2, \ldots$$

- τ is a discrete time stopping time for filtration $\{\mathcal{F}_n\}$ if and only if

$$\{\tau = n\} \in \mathcal{F}_n \quad \text{for all } n = 0, 1, 2, \ldots$$

- A random time τ is a stopping time if at any point in time you can determine whether or not the time τ has already occurred based on the information currently available

Discrete Time Stopping Time: Examples

- Any non-random time is a stopping time
- *Hitting time*: let $\{X_n\}$ be a process adapted to $\{\mathcal{F}_n\}$. For a Borel set B, define a random variable

$$\tau(\omega) = \inf\{n \geq 0 : X_n(\omega) \in B\}$$

  with $\tau(\omega) = \infty$ if $X_n \notin B$ for all n. Then τ is a stopping time for $\{\mathcal{F}_n\}$
- Hitting time is the *first* time (if ever) the process takes a value in the set B
- If θ, τ, τ_1, τ_2,. . . are all stopping times for the same filtration then the following are also stopping times:

$$\min(\theta, \tau),\ \max(\theta, \tau),\ \theta + \tau,\ \sup_{n \geq 1} \tau_n,\ \inf_{n \geq 1} \tau_n$$

Optional Stopping Theorem: Motivation

- Suppose $\{(X_n, \mathcal{F}_n)\}$ is a martingale
- Then $\mathbb{E}(X_n) = \mathbb{E}(X_0)$ for any n
- If τ is a stopping time for $\{\mathcal{F}_n\}$, is it true that

$$\mathbb{E}(X_\tau) = \mathbb{E}(X_0)?$$

- Equality not true in general
- Optional stopping theorem provides sufficient conditions for above equality to hold
- Can use martingales and optional stopping theorem to obtain distributional properties of τ

Stopped Process

- Suppose $\{X_n\}$ is a SP and τ is a stopping time for filtration $\{\mathcal{F}_n\}$
- The **stopped (at time τ) process** $\{X_{n\wedge\tau}, n = 0, 1, 2, \ldots\}$ is defined by

$$X_{n\wedge\tau} = \begin{cases} X_n & n \le \tau \\ X_\tau & n > \tau \end{cases}$$

- If $\{(X_n, \mathcal{F}_n)\}$ is a sub-martingale and τ is an $\{\mathcal{F}_n\}$-stopping time then $\{(X_{n\wedge\tau}, \mathcal{F}_n)\}$ is a sub-martingale. In particular, $\mathbb{E}(X_{n\wedge\tau}) \ge \mathbb{E}(X_0)$.
- If $\{(X_n, \mathcal{F}_n)\}$ is a martingale and τ is an $\{\mathcal{F}_n\}$-stopping time then $\{(X_{n\wedge\tau}, \mathcal{F}_n)\}$ is a martingale. In particular, $\mathbb{E}(X_{n\wedge\tau}) = \mathbb{E}(X_0)$.

Doob's Optional Stopping Theorem

- Suppose $\{(X_n, \mathcal{F}_n)\}$ is a sub-martingale and τ is an $\{\mathcal{F}_n\}$-stopping time. If
  - $\tau < \infty$ a.s., and
  - $\{X_{n\wedge\tau}, n = 0, 1, 2, \ldots\}$ is uniformly integrable (2)
- Then $\mathbb{E}(X_\tau) \geq \mathbb{E}(X_0)$.
- If instead $\{(X_n, \mathcal{F}_n)\}$ is a martingale then $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.
- If any of the following is true then (2) is satisfied
  - τ is a bounded stopping time: $\tau \leq N$ a.s. for some constant $N < \infty$
  - There is a constant $c < \infty$ such that $\mathbb{E}|X_n| \leq c$ for all n

Gambler's Ruin: Probability of Ruin

- Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = 1/2$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} \xi_k$.
- Fix integers $a, b > 0$.
- Define $\tau = \inf\{n \geq 0 : S_n = b$ or $S_n = -a\}$
- Then:
  - τ is a stopping time with $\tau < \infty$ a.s.
  - $\{S_{n \wedge \tau}\}$ is uniformly integrable
  - Probability of ruin

$$\mathbb{P}(S_\tau = -a) = \frac{b}{a+b}$$

Gambler's Ruin: Expected Number of Games

- Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $I\!\!P(\xi_k = 1) = I\!\!P(\xi_k = -1) = 1/2$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} \xi_k$.
- Fix integers $a, b > 0$.
- Define $\tau = \inf\{n \geq 0 : S_n = b$ or $S_n = -a\}$
- Let $Y_n = S_n^2 - n$
- Then:
  - τ is a stopping time with $\tau < \infty$ a.s.
  - $\{Y_{n \wedge \tau}\}$ is uniformly integrable
  - Expected number of games played:

$$I\!\!E(\tau) = ab$$