

## Last Time

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- Discrete time sub- and super-martingales

Today's lecture: Sections 4.2

## Happy Halloween!

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What do you get when you divide the circumference of a jack-o-lantern by its diameter?

Happy Halloween!

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What do you get when you divide the circumference of a jack-o-lantern by its diameter?

Pumpkin  $\pi$

## Continuous Time Filtration & Adapted Process

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- A continuous time **filtration** is a nondecreasing family  $\{\mathcal{F}_t, t \geq 0\}$  of sub- $\sigma$ -fields of  $\mathcal{F}$ , i.e. each  $\mathcal{F}_t$  is a  $\sigma$ -field and

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad \text{for all } 0 \leq s \leq t$$

- A continuous time process  $\{X_t, t \geq 0\}$  is **adapted** to filtration  $\{\mathcal{F}_t, t \geq 0\}$  if for all  $t \geq 0$  the random variable  $X_t$  is measurable with respect to the  $\sigma$ -field  $\mathcal{F}_t$
- The **canonical filtration** of a continuous time SP  $\{X_t, t \geq 0\}$  is the filtration  $\{\mathcal{F}_t^X, t \geq 0\}$ , where  $\mathcal{F}_t^X = \sigma(X_u : 0 \leq u \leq t)$
- The canonical filtration is the smallest filtration to which  $\{X_t\}$  is adapted

# Continuous Time Martingale

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- A continuous time **martingale** is a pair  $\{(X_t, \mathcal{F}_t), t \geq 0\}$  which satisfies:
  - $\{\mathcal{F}_t\}$  is a filtration
  - $\{X_t\}$  is adapted to  $\{\mathcal{F}_t\}$
  - $\mathbb{E}|X_t| < \infty$  for all  $t \geq 0$
  - $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$  for all  $0 \leq s \leq t$  (1)
- The pair  $\{(X_t, \mathcal{F}_t), t \geq 0\}$  is a continuous time **sub-martingale** if (1) above is replaced by

$$\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s \quad \text{for all } 0 \leq s \leq t$$

- The pair  $\{(X_t, \mathcal{F}_t), t \geq 0\}$  is a continuous time **super-martingale** if (1) above is replaced by

$$\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s \quad \text{for all } 0 \leq s \leq t$$

## Basic Properties

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- For martingales:

$$\mathbb{E}(X_t) = \mathbb{E}(X_0) \text{ for all } t \geq 0$$

- For sub-martingales:

$$\mathbb{E}(X_t) \geq \mathbb{E}(X_0) \text{ for all } t \geq 0$$

- For super-martingales:

$$\mathbb{E}(X_t) \leq \mathbb{E}(X_0) \text{ for all } t \geq 0$$

- If  $\{(X_t, \mathcal{F}_t)\}$  is a martingale,  $\{\mathcal{G}_t\}$  is a filtration,  $\mathcal{G}_t \subseteq \mathcal{F}_t$ , and  $\{X_t\}$  is  $\{\mathcal{G}_t\}$ -adapted, then  $\{(X_t, \mathcal{G}_t)\}$  is a martingale
  - Similar statements hold for sub-martingales and super-martingales

## Zero-Mean, Independent Increment Processes

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- If  $\{X_t\}$  is adapted to filtration  $\{\mathcal{F}_t\}$  with  $\mathbb{E}|X_t| < \infty$  and for all  $0 \leq s \leq t$ 
  - $\mathbb{E}(X_t - X_s) = 0$
  - $X_t - X_s$  is independent  $\mathcal{F}_s$
- Then  $\{(X_t, \mathcal{F}_t)\}$  is a martingale
- **Corollary:** if  $\{W_t\}$  is a Brownian motion with canonical filtration  $\{\mathcal{F}_t^W\}$  then  $\{(W_t, \mathcal{F}_t^W)\}$  is a martingale

## Functions of Sub-Martingales

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- Suppose  $\{(X_t, \mathcal{F}_t)\}$  is a martingale and  $g$  is a convex function such that  $\mathbb{E}|g(X_t)| < \infty$  for all  $t \geq 0$ . Then  $\{(g(X_t), \mathcal{F}_t)\}$  is a sub-martingale
- Suppose  $\{(Z_t, \mathcal{G}_t)\}$  is a sub-martingale and  $h$  is a nondecreasing, convex function such that  $\mathbb{E}|h(Z_t)| < \infty$  for all  $t \geq 0$ . Then  $\{(h(Z_t), \mathcal{G}_t)\}$  is a sub-martingale

## Right-Continuous Filtrations

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- Suppose  $\{\mathcal{F}_t\}$  is a filtration and define

$$\mathcal{F}_{t+} \doteq \bigcap_{h>0} \mathcal{F}_{t+h}$$

- $\mathcal{F}_{t+}$  is “information available immediately after time  $t$ ”
- A filtration is **right-continuous** if  $\mathcal{F}_{t+} = \mathcal{F}_t$  for all  $t \geq 0$
- “Each new piece of information has a definite first time of arrival”

## Right-Continuous Filtrations (cont.)

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- Not every filtration is right-continuous (see Example 4.2.12)
- However, a filtration can be *augmented* so it is right-continuous
- *For any filtration  $\{\mathcal{F}_t\}$  we will always assume the “usual conditions”*
  - $\{\mathcal{F}_t\}$  is right-continuous
  - $N \in \mathcal{F}_0$  whenever “ $\mathbb{P}(N) = 0$ ”
- Consequence: if  $\{(X_t, \mathcal{F}_t)\}$  is a martingale and  $\{\mathcal{F}_t\}$  is right-continuous then  $\{X_t\}$  has an RCLL modification