Last Time

- Filtration
- Adapted process
- Martingale

Today's lecture: Sections 4.1.1, 4.1.2, 4.1.3

Square Integrable Martingales

- A **square integrable martingale** is a martingale with $\mathbb{E}(X_n^2) < \infty$ for all n
- A square integrable SP is a martingale if and only if it has a zero-mean, orthogonal difference sequence
- That is $\{X_n\}$ is a square integrable martingale if and only if its differences $D_n = X_n - X_{n-1}$ satisfy $\mathbb{E}(D_n) = 0$ and

$$\mathbb{E}[D_{n+1}|\sigma(D_0, D_1, \ldots, D_n)] = \mathbb{E}[D_{n+1}],$$

  where the conditional expectation is interpreted in the L^2 sense

Sub-Martingales

- A discrete time **sub-martingale** is a pair $\{(X_n, \mathcal{F}_n)\}$ which satisfies:
  - $\mathbb{E}|X_n| < \infty$ for all n
  - $\{\mathcal{F}_n\}$ is a filtration
  - $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$
  - **$\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$ for all n**
- If $\{X_n\}$ is a sub-martingale, then

$$\mathbb{E}(X_0) \leq \mathbb{E}(X_1) \leq \mathbb{E}(X_2) \leq \cdots$$

Super-Martingales

- A discrete time **super-martingale** is a pair $\{(X_n, \mathcal{F}_n)\}$ which satisfies:
  - $\mathbb{E}|X_n| < \infty$ for all n
  - $\{\mathcal{F}_n\}$ is a filtration
  - $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$
  - $\boldsymbol{\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n}$ **for all** $\boldsymbol{n}$
- If $\{X_n\}$ is a super-martingale, then

$$\mathbb{E}(X_0) \geq \mathbb{E}(X_1) \geq \mathbb{E}(X_2) \geq \cdots$$

Example 1: Sums of Independent RV's

- Let ξ_k be a sequence of independent RV's with $\mathbb{E}|\xi_k| < \infty$ for all k
- Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} \xi_k$
- If $\mathbb{E}(\xi_k) = 0$ then $\{S_n\}$ is a martingale
- If $\mathbb{E}(\xi_k) \geq 0$ then $\{S_n\}$ is a sub-martingale
- If $\mathbb{E}(\xi_k) \leq 0$ then $\{S_n\}$ is a super-martingale

Example 3: Products of Independent, Positive RV's

- Let Y_k be a sequence of independent RV's with $\mathbb{E}|Y_k| < \infty$ and $\mathbb{P}(Y_k > 0) = 1$ for all k
- Let $M_0 = 1$ and $M_n = \prod_{k=1}^{n} Y_k$
- If $\mathbb{E}(Y_k) = 1$ then $\{M_n\}$ is a martingale
- If $\mathbb{E}(Y_k) \geq 1$ then $\{M_n\}$ is a sub-martingale
- If $\mathbb{E}(Y_k) \leq 1$ then $\{M_n\}$ is a super-martingale

Functions of Sub-Martingales

- Suppose $\{(X_n, \mathcal{F}_n)\}$ is a martingale and g is a convex function such that $\mathbb{E}|g(X_n)| < \infty$ for all n. Then $\{(g(X_n), \mathcal{F}_n)\}$ is a sub-martingale
- Suppose $\{(Z_n, \mathcal{G}_n)\}$ is a sub-martingale and h is a nondecreasing, convex function such that $\mathbb{E}|h(Z_n)| < \infty$ for all n. Then $\{(h(Z_n), \mathcal{G}_n)\}$ is a sub-martingale