

Last Time

- Brownian motion defined
- Existence of BM
- Continuity of sample paths of BM
- Basic properties of BM
- Relatives of BM (BM with drift, Geometric BM, O-U process)

Today's lecture: Sections 4.1.1, 4.1.2

Discrete Time Filtration

- A discrete time **filtration** on a measurable space (Ω, \mathcal{F}) is a nondecreasing family $\{\mathcal{F}_n, n = 0, 1, 2, \dots\}$ of sub- σ -fields of \mathcal{F} , i.e. each \mathcal{F}_n is a σ -field and

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}$$

- A filtration represents the accumulation of information over time

Adapted Process

- A discrete time SP $\{X_n, n = 0, 1, \dots\}$ is **adapted** to the filtration $\{\mathcal{F}_n, n = 0, 1, \dots\}$ if for all n , X_n is \mathcal{F}_n -measurable
- The **canonical filtration** (aka filtration generated by) a discrete time SP $\{X_n, n = 0, 1, \dots\}$ is the filtration $\{\mathcal{F}_n^X, n = 0, 1, \dots\}$, where $\mathcal{F}_n^X = \sigma(X_0, X_1, \dots, X_n)$
- The canonical filtration is the smallest filtration to which $\{X_n\}$ is adapted
- Example: for two coin tosses ($\Omega = \{HH, HT, TH, TT\}$ and $\mathcal{F} = 2^\Omega$), let X_n be the RV representing the total number of heads observed by time $n = 0, 1, 2$. Then

$$\mathcal{F}_0^X = \sigma(X_0) = \{\emptyset, \Omega\}$$

$$\mathcal{F}_1^X = \sigma(X_0, X_1) = \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\}$$

$$\mathcal{F}_2^X = \sigma(X_0, X_1, X_2) = \mathcal{F}$$

Discrete Time Martingale

- A discrete time **martingale** is a pair $\{(X_n, \mathcal{F}_n)\}$ which satisfies:
 - $\{\mathcal{F}_n\}$ is a filtration
 - $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$
 - $\mathbb{E}|X_n| < \infty$ for all n
 - $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ for all n
- A SP $\{X_n\}$ is often called a martingale if the pair $\{(X_n, \mathcal{F}_n^X)\}$, where $\{\mathcal{F}_n^X\}$ is the SP's canonical filtration, is a martingale according to the above definition

Gambling Interpretation

A martingale represents a fair game

- A gambler plays a game a number of times. Consider the adapted sequence $\{(X_n, \mathcal{F}_n)\}$, where
 - X_0 = initial wealth of gambler
 - X_n = wealth of gambler after n -th play
 - \mathcal{F}_n = information available to gambler after n -th play
- The condition $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ implies that the game is fair

Basic Properties of Martingales

If $\{(X_n, \mathcal{F}_n)\}$ is a martingale then

- $\mathbb{E}(X_{n+k} | \mathcal{F}_n) = X_n$ for all $k = 1, 2, \dots$
- $\mathbb{E}(X_n) = \mathbb{E}(X_0)$ for all $n = 1, 2, \dots$
- If $\{\mathcal{G}_n\}$ is a filtration with $\mathcal{G}_n \subseteq \mathcal{F}_n$ for all $n = 0, 1, \dots$ and $\{X_n\}$ is adapted to $\{\mathcal{G}_n\}$, then $\{(X_n, \mathcal{G}_n)\}$ is a martingale
 - In particular, $\{(X_n, \mathcal{F}_n^X)\}$ is a martingale

Example 1: Sums of Independent, Mean 0 RV's

- Let ξ_k be a sequence of independent RV's with $\mathbb{E}|\xi_k| < \infty$ and $\mathbb{E}(\xi_k) = 0$ for all k
- Let $S_0 = 0$ and $S_n = \sum_{k=1}^n \xi_k$
- Then $\{(S_n, \mathcal{F}_n^S)\}$ is a martingale
- Letting $\mathcal{F}_n^\xi = \sigma(\xi_1, \dots, \xi_n)$, the pair $\{(S_n, \mathcal{F}_n^\xi)\}$ is also a martingale

Martingale Differences

- Suppose $X_n = \sum_{i=0}^n D_i$ for all $n = 0, 1, 2, \dots$
- That is, $D_n = X_n - X_{n-1}$
- Note that $\mathcal{F}_n^X = \mathcal{F}_n^D$ for all n
- For any filtration $\{\mathcal{F}_n\}$, $\{(X_n, \mathcal{F}_n)\}$ is a martingale if and only if
 - $\mathbb{E}|D_n| < \infty$ for all n
 - $\{D_n\}$ is adapted to $\{\mathcal{F}_n\}$
 - $\mathbb{E}(D_{n+1}|\mathcal{F}_n) = 0$

Example 2: Martingale Transform

- Let ξ_k be a sequence of independent RV's with $\mathbb{E}|\xi_k| < \infty$ and $\mathbb{E}(\xi_k) = 0$ for all k
- Let $\{V_n\}$ be a SP with $|V_n| \leq C$ for some $C < \infty$ and all n
- Let $Y_n = \sum_{k=1}^n V_k \xi_k$.
- Assume that ξ_{n+1} is independent of $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n, V_1, \dots, V_{n+1})$.
- Then $\{(Y_n, \mathcal{F}_n)\}$ is a martingale

Martingale Transforms

- A SP $\{V_n\}$ is **previsible** (or predictable) with respect to a filtration $\{\mathcal{F}_n\}$ if for all n , V_{n+1} is \mathcal{F}_n -measurable
- Suppose $\{(X_n, \mathcal{F}_n)\}$ is a martingale and $\{V_n\}$ is a previsible sequence with respect to $\{\mathcal{F}_n\}$.
- The **martingale transform of V with respect to X** is the SP $\{Y_n\}$, where

$$Y_n = \sum_{k=1}^n V_k (X_k - X_{k-1})$$

- If $\mathbb{E}(V_n^2) < \infty$ and $\mathbb{E}(X_n^2) < \infty$ for all n then $\{(Y_n, \mathcal{F}_n)\}$ is a martingale

Example 3: Products of Independent, Positive RV's

- Let Y_k be a sequence of independent RV's with $\mathbb{E}|Y_k| < \infty$, $\mathbb{E}(Y_k) = 1$, and $\mathbb{P}(Y_k > 0) = 1$ for all k
- Let $M_0 = 1$ and $M_n = \prod_{k=1}^n Y_k$
- Then $\{(M_n, \mathcal{F}_n^M)\}$ is a martingale
- The pair $\{(M_n, \mathcal{F}_n^Y)\}$ is also a martingale

Example 4: Poyla's Urn Scheme

- Consider an urn that contains red and green balls
- At time 0 there is one ball of each color
- At time n randomly select a ball. Then return it to the urn and add one ball with the same color as the selected ball.
- Let X_n be the fraction of red balls at time n (after selecting and adding a ball)
- Then $\{(X_n, \mathcal{F}_n^X)\}$ is a martingale

Midterm Grades

- Mean = ; Median =
- 90+:
- 80-89:
- 70-79:
- 60-69:
- 50-59:
- 40-49:
- <40: