Last Time

- Random vectors
- Characteristic function
- Gaussian random vectors
- Gaussian stochastic process

Today's lecture: Sections 3.2, 3.3

Notes on Exercise 3.2.12

- Consider independent RVs X and S where $X \sim N(0,1)$ and $\mathbb{P}(S=1) = \mathbb{P}(S=-1) = 1/2$
- Let $Y = SX$
- Then $Y \sim N(0,1)$ and $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$
- BUT X and Y are NOT independent
- (X, Y) is NOT a Gaussian random vector
- $X + Y$ is NOT a Gaussian random variable

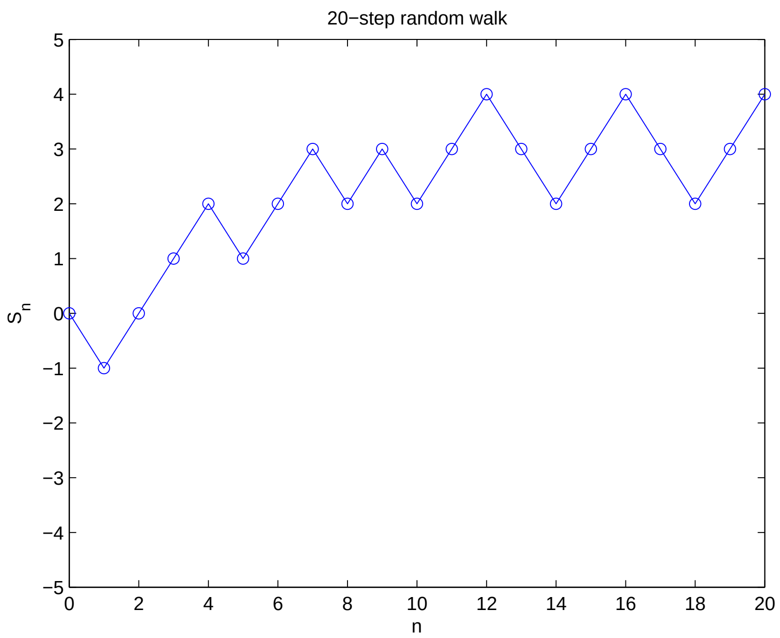
Key Properties of Stochastic Processes

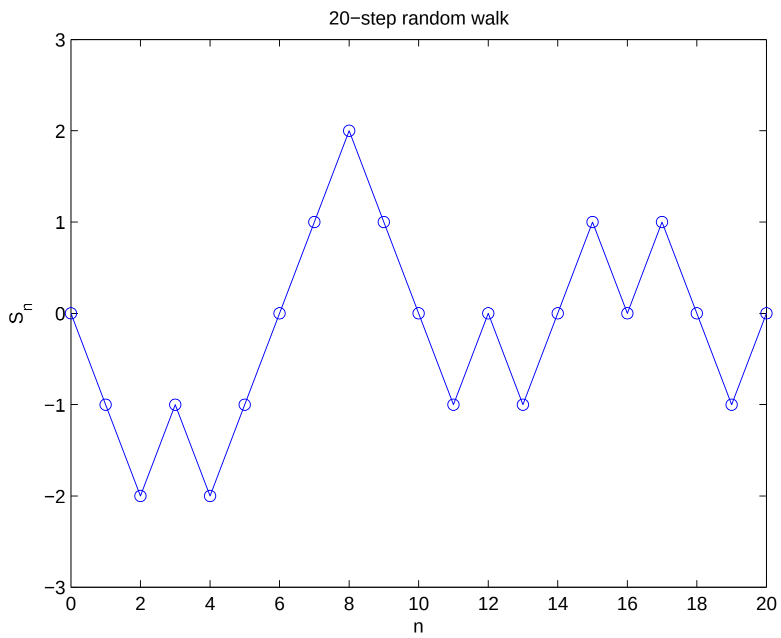
- Distributional properties
  - Gaussian
  - Stationary
- Dependence structures
  - Martingale
  - Markov
- Sample path properties
  - Continuous or RCLL
- Main examples: Brownian motion, Poisson process

Random Walk

- Let $\xi_1, \xi_2, \ldots$ be an i.i.d. sequence of RV's. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} \xi_k$.
- The discrete time process $\{S_n, n = 0, 1, 2, \ldots\}$ is called a **random walk**.
- If $\mathbb{E}(\xi_k) = 0$ the random walk is *symmetric*.
- If ξ_k only takes values -1 or 1, the random walk is *simple*.

Example: Random Walk Sample Paths





Convergence of Scaled Random Walks

- Let $\xi_1, \xi_2, \ldots$ be an i.i.d. sequence of RV's with $\mathbb{E}(\xi_k) = 0$ and $\mathbb{E}(\xi_k^2) = 1$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^n \xi_k$.
- CLT implies:

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{L}} N(0,1) \text{ as } n \to \infty$$

- Fix $t \geq 0$. Then

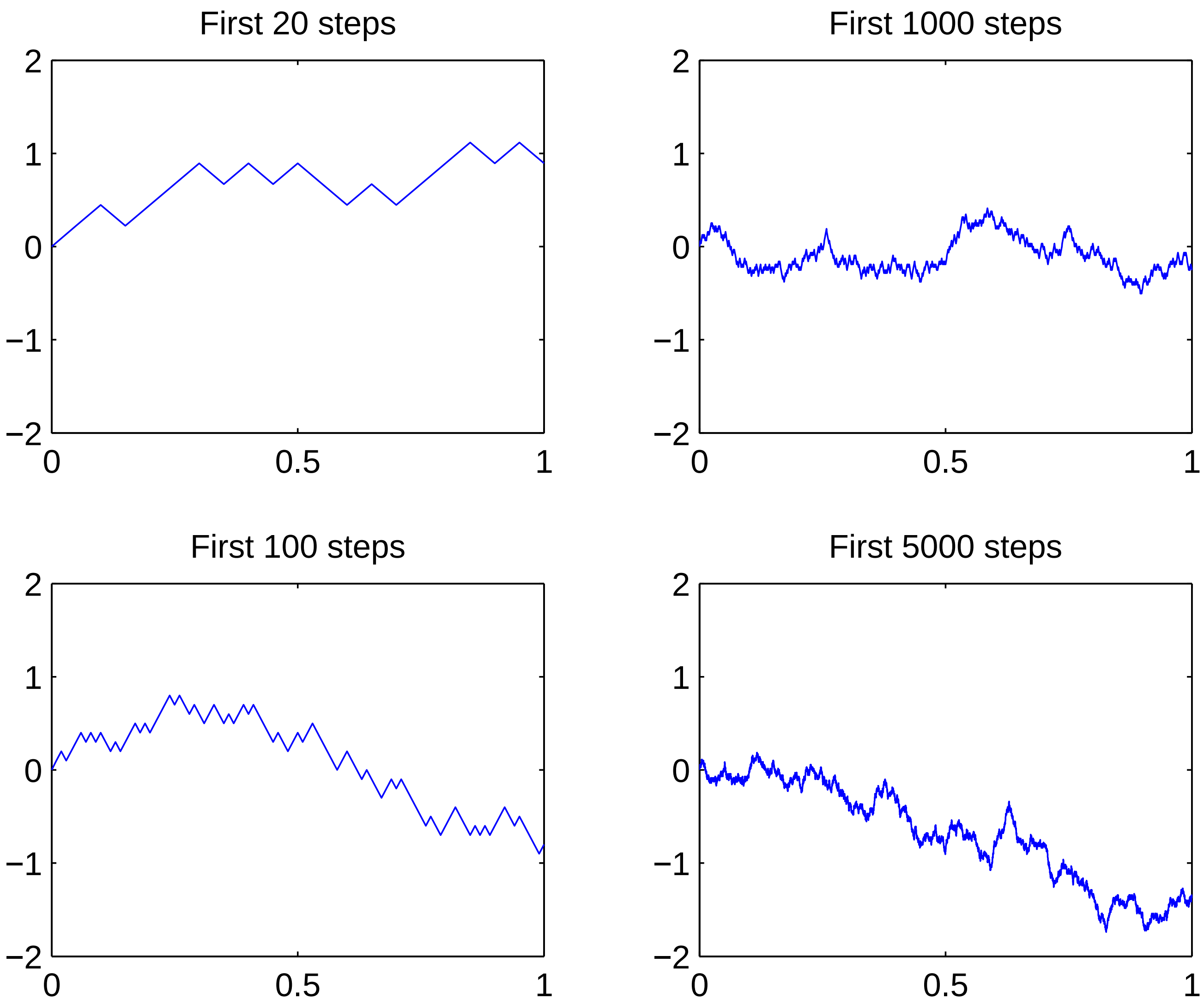
$$\frac{S_{[nt]}}{\sqrt{n}} \xrightarrow{\mathcal{L}} N(0,t) \text{ as } n \to \infty$$

- For each n define a SP

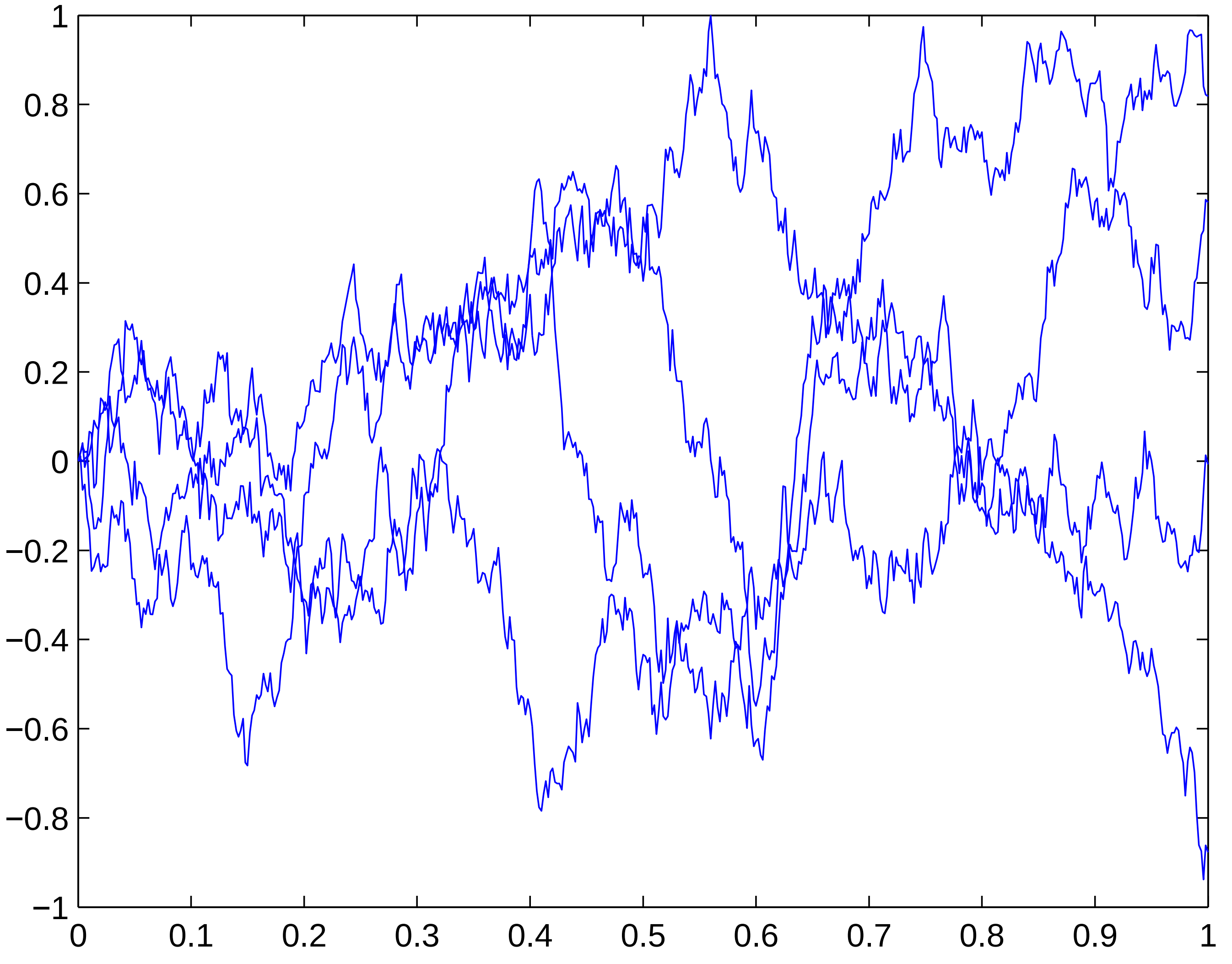
$$S_t^{(n)} \doteq \frac{1}{\sqrt{n}} \Big(S_{[nt]} + (nt - [nt]) \xi_{[nt]+1} \Big), \quad t \geq 0$$

As $n \to \infty$, $S^{(n)}$ "converges" to Brownian Motion

Convergence of Scaled Random Walks



Example: Brownian Motion Sample Paths



Stationary Processes

- A SP $\{X_t, t \geq 0\}$ is **strongly stationary** if for all integers $n < \infty$, times $t_1, \ldots, t_n \geq 0$, and any $h > 0$

$$(X_{t_1}, \ldots X_{t_n}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_n+h})$$

  That is, its FDD's are invariant under time shifts

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{t_1+h,\ldots,t_n+h}(x_1, \ldots, x_n)$$

- A SP $\{X_t, t \geq 0\}$ is **weakly stationary** if:

$$\begin{aligned} \mu(t) &\doteq \mathbb{E}(X_t) = \mu, \\ \rho(t,s) &\doteq \mathbb{E}[(X_t - \mu(t))(X_s - \mu(s))] = r(|t-s|), \end{aligned}$$

  where μ is a constant and r is a function only of the time difference $|t-s|$

Gaussian Stationary Processes

- A Gaussian SP $\{X_t, t \geq 0\}$ is strongly stationary if and only if it is weakly stationary

Stationary Increment Processes

- A SP $\{X_t, t \geq 0\}$ has **stationary increments** if the distribution of the RV $X_t - X_s$ depends on t and s only through their difference $|t - s|$
- That is, for all $h > 0$

$$X_{t+h} - X_{s+h} \stackrel{d}{=} X_t - X_s$$

- If a process is stationary then it has stationary increments
- But a stationary increment process need not be stationary

Continuous in Probability

- A SP $\{X_t, t \in \mathcal{I}\}$ is **continuous in probability** if for each $t_0 \in \mathcal{I}$
$$\lim_{t \to t_0} \mathbb{P}(|X_t - X_{t_0}| > \epsilon) = 0 \text{ for all } \epsilon > 0$$
- Only FDD's are needed to determine if a process is continuous in probability
- But continuity in probability does not imply that the paths are continuous

Continuous and RCLL Sample Paths

- A SP $\{X_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, I\!P)$ has **continuous sample paths** (w.p. 1) if the set

$$\{\omega : t \mapsto X_t(\omega) \text{ is continuous}\}$$

has probability 1.

- A SP $\{X_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, I\!P)$ has **right-continuous, left limit (RCLL) sample paths** (w.p. 1) if the set of ω such that:

$$\text{(RC)} \quad \lim_{h \downarrow 0} X_{t+h}(\omega) = X_t(\omega) \text{ for all } t \geq 0$$

$$\text{(LL)} \quad \lim_{h \downarrow 0} X_{t-h}(\omega) \text{ exists}$$

has probability 1.

Kolmogorov's Continuity Criteria

- Let $\{X_t, 0 \le t \le T\}$ be a SP
- If there exist *positive* constants α, β, and C such that

$$\mathbb{E}(|X_t - X_s|^\alpha) \le C|t-s|^{1+\beta}, \text{ for all } 0 \le s,t \le T$$

- Then there exists a modification of $\{X_t\}$ that has continuous sample paths (w.p. 1)
- Also true for $\{X_t, t \ge 0\}$

Remark on Hölder Continuity

- A function $t \mapsto f(t)$ is **Hölder continuous** with exponent γ if there exists a positive constant C such that

$$|f(t) - f(s)| \leq C|t-s|^{\gamma} \text{ for all } 0 \leq s, t \leq T$$

- Special case ($\gamma = 1$): A function $t \mapsto f(t)$ is **Lipschitz continuous** if there exists a positive constant C such that

$$|f(t) - f(s)| \leq C|t-s| \text{ for all } 0 \leq s, t \leq T$$

- If a function is Hölder continuous with exponent $\gamma > 0$ then it is continuous.
- The constants α and β in the Kolmogorov continuity criteria determine the smoothness of the continuous modification