

## Last Time

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- Special Cases of CE
- Properties of CE
- Convergence of CE's
- Tower property
- Taking out what is known
- Conditional probability

Today's lecture: Section 3.1

## Definition of Stochastic Process

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- A **stochastic process** is a collection of random variables on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ :  $\{X_t : t \in \mathcal{I}\}$  for some index set  $\mathcal{I}$
- Discrete time SP:  $\mathcal{I}$  is countable, e.g.

$$\{X_n : n = 0, 1, 2, \dots, N\} \text{ or } \{X_n : n = 0, 1, 2, \dots\}$$

- Continuous time SP:  $\mathcal{I}$  is uncountable, e.g.

$$\{X_t : 0 \leq t \leq T\} \text{ or } \{X_t : t \geq 0\}$$

## Some Questions of Interest

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- What is the distribution of values of the process at any given times?
- How do future values of the process depend on past values or information?
- Does the process “converge” in the long run?
- Does the process move continuously in time or does it have jumps? Is it nondecreasing?
- When is the first time the process reaches some level?
- What is the maximum value of the process over some time interval?

## Why Continuous Time?

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- High frequency observations
- Limits of discrete time models
- Continuous time models often more tractable than discrete time models
- In Finance: no arbitrage theory easier in continuous time

## Two Ways of Thinking about a SP

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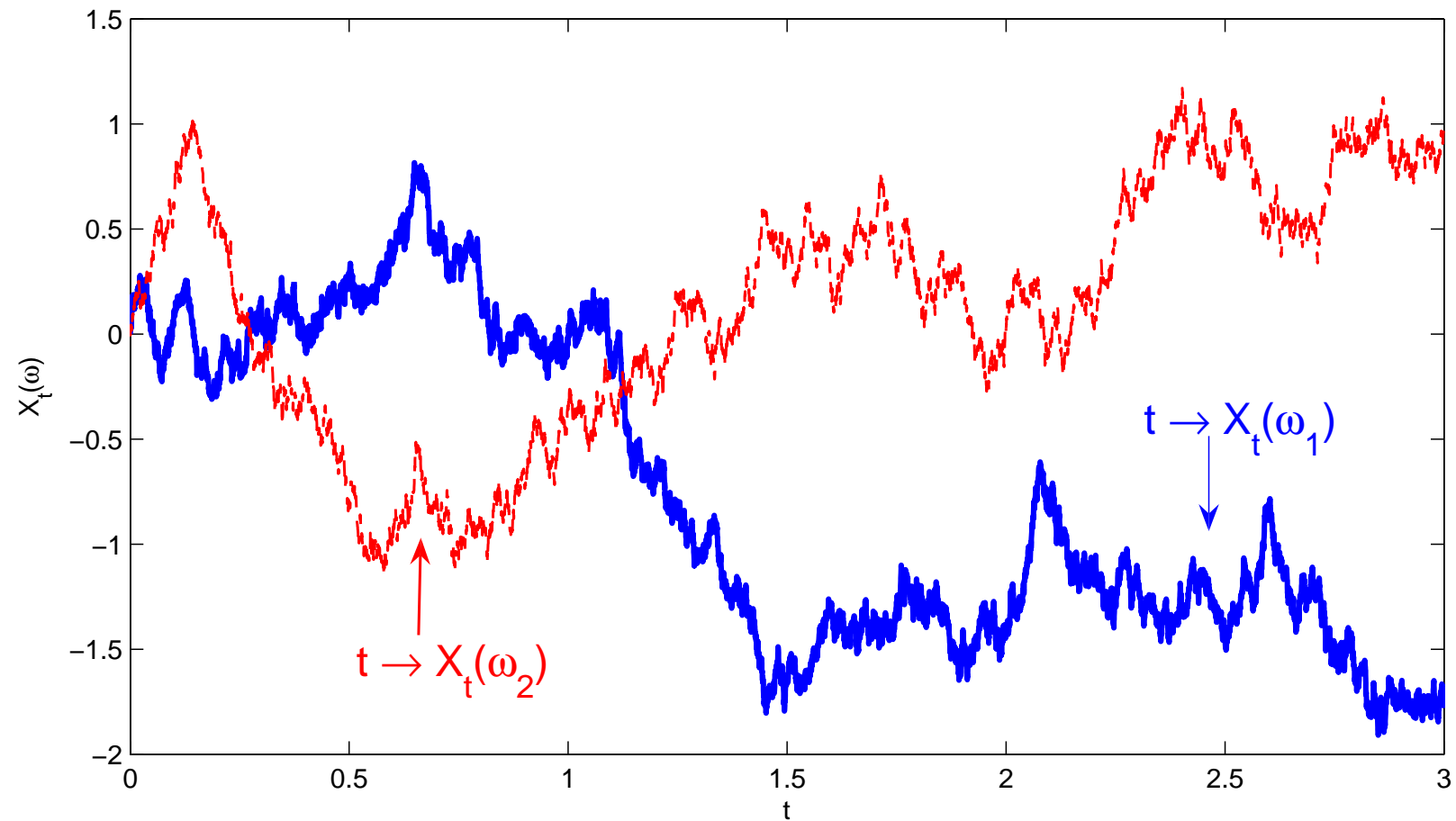
- For fixed  $t$ ,  $X_t$  is a random variable, i.e.

$$X_t(\cdot) : \Omega \rightarrow \mathbb{R}$$

- For fixed  $\omega$ , the map  $t \mapsto X_t(\omega)$  is called the **sample path** (aka trajectory) associated with  $\omega$

# Illustration: Sample Paths

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## Finite Dimensional Distributions

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- For any integer  $n < \infty$  and times  $t_1, t_2, \dots, t_n \in \mathcal{I}$ , the joint distribution of the random vector  $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$  is denoted by  $F_{t_1, t_2, \dots, t_n}$  and given by

$$F_{t_1, t_2, \dots, t_n}(x_1, x_2, \dots, x_n) = \mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_n} \leq x_n)$$

for all  $x_1, x_2, \dots, x_n \in \mathbb{R}$

- The collection of functions

$$\{F_{t_1, t_2, \dots, t_n} : n < \infty \text{ and } t_1, \dots, t_n \in \mathcal{I}\},$$

is called the **finite dimensional distributions (FDD'S)** of the stochastic process  $X$ .

## Existence of Processes

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- A collection of distribution functions is **consistent** if

$$\begin{aligned} & \lim_{x_k \rightarrow \infty} F_{t_1, \dots, t_n}(x_1, \dots, x_n) \\ &= F_{t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_n}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n), \end{aligned}$$

for all integers  $n$ , times  $t_1, \dots, t_n \in \mathcal{I}$ ,  $x_1, \dots, x_n \in \mathbb{R}$ , and any integer  $1 \leq k \leq n$

- If  $X$  is a SP then its FDD's are consistent
- Given any consistent collection of FDD's, there exists a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a stochastic process  $X$  defined on it such that  $X$  has the specified FDD's

## Example 3.1.6

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- Consider  $\Omega = [0, 1]$ , with its Borel  $\sigma$ -field, and the uniform probability measure. Let  $\mathcal{I} = [0, 1]$ .
- Define processes  $Y_t(\omega) = 0$  for all  $\omega \in \Omega, t \in \mathcal{I}$  and

$$X_t(\omega) = \begin{cases} 1, & t = \omega, \\ 0, & \text{otherwise} \end{cases}$$

- Main point: FDD's are not enough to determine sample path properties of the process

## Versions and Modifications

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- Two SP's  $X$  and  $Y$  are **versions** of one another if they have the same FDD's.
- Two SP's  $X$  and  $Y$ , defined on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , are **modifications** of one another if

$$\mathbb{P}(X_t = Y_t) = 1 \text{ for all } t \in \mathcal{I}$$

- If  $X$  and  $Y$  are modifications, then they are also versions of one another
- Two versions do not have to be defined on the same probability space

## $\sigma$ -field generated by a SP

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- The  $\sigma$ -field generated by a SP  $X$ , denoted  $\mathcal{F}_X$  is the smallest  $\sigma$ -field containing  $\sigma(X_t)$  for all  $t \in \mathcal{I}$ .
- Information contained in the whole process
- Two stochastic processes  $X$  and  $Y$  are independent if their generated  $\sigma$ -fields  $\mathcal{F}_X$  and  $\mathcal{F}_Y$  are independent