Euler characteristic densities, Gaussian volumes of tubes and correlated conjunctions

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January 2, 2002

Abstract

In this work we describe some new results in approximating the distribution of the maximum of a smooth stochastic process on a manifold M, specifically Gaussian and closely related processes. We use the expected Euler characteristic method to approximate this distribution and describe a new formula, a Gaussian version of the classical Kinematic Fundamental Formulae of integral geometry. This result relates the Euler characteristic densities of a certain class of processes to coefficients in certain power series expansions of the standard Gaussian measure of certain tubes.

We give some simple applications of the result, including a simple derivation of the EC densities of Gaussian and χ^2 processes. Although these two results are not new, the Gaussian KFF sheds some light on them gives a geometric interpretation of them. As a corollary to the result for χ^2 processes, the Gaussian KFF yields the EC densities of non-central χ^2 processes, which have not been published previously.

This Gaussian KFF also shows how to use the EC approach for certain fields with piecewise smooth level sets. As an application, we derive the EC densities of the process given by taking the pointwise minimum of two i.i.d. Gaussian processes, known as a correlated conjunction. To validate these approximations, we conclude with the results of a small simultation study.

KEY WORDS: Random fields, Gaussian processes, manifolds, Euler characteristic, excursions, Riemannian geometry.
AMS SUBJECT CLASSIFICATIONS: Primary 60G15, 60G60, 53A17 58A05; Secondary 60G17, 62M40, 60G70

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1 Introduction

In their recent work, Takemura and Kuriki [11, 10] have used the volume of tubes to approximate the distribution of the maximum of a class of centered, unit-variance Gaussian processes, the class of finite Karhunen-Loève expansion Gaussian processes. Processes of this form have as parameter space a manifold (or piecewise smooth manifold) $M \subset S(\mathbb{R}^n)$, the unit sphere in $\mathbb{R}^n$, and are defined by

$$f(p, \omega) = \langle p, \xi(\omega) \rangle \tag{1}$$

where $\xi \sim N(0, I_{n \times n})$. The resulting approximations have the form

$$\mathrm{P}\left[\sup_{p \in M} f(p) \geq u\right] \simeq \sum_{i=0}^{n} c_i \mathcal{L}_i^1(M) Q_i(u) \tag{2}$$

where $\mathcal{L}_i^1(M)$ are geometric invariants of M related to the second fundamental form of M in $S(\mathbb{R}^n)$ and $Q_i(u)$ are quantities related to the tails of the distribution of χ_i^2 random variables and c_i are known, universal constants. The precise definitions of the $\mathcal{L}_i^1(M)$'s for manifolds without boundary are given below in (3), from which it can be seen that the $\mathcal{L}_i^1(M)$'s are intrinsic invariants of M. To simplify formulas we will only give formulas when M is a smooth manifold without boundary. In this case, for $\lambda \in \mathbb{R}$, we define

$$\mathcal{L}_j^\lambda(M) = \begin{cases} \frac{1}{(q-j)!} \int_M \mathrm{Tr}^M \left(\left(-R + \frac{\lambda}{2} I^2 \right)^{(q-j)/2} \right) d\mathrm{Vol}_M & q - j > 0 \text{ even} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

with Vol_M representing Riemannian measure on M, R the curvature tensor of M and Tr^M the trace on the double differential forms of M, for further details, the reader is referred to [6, 12, 14].

Perhaps a more useful characterization of the constants $\mathcal{L}_i^1(M)$, $1 \leq i \leq n$, indeed the characterization used by Takemura and Kuriki, is that they are coefficients in the expansion for the spherical volume of a tube of given radius around M. Specifically, for ρ small (see below)

$$\mathrm{Vol}_{S(\mathbb{R}^n)}\left(T(M, \rho)\right) = \sum_{j=0}^{n} \mathcal{L}_i^1(M) \frac{2\pi^{(n-i)/2}}{\Gamma\left(\frac{n-i}{2}\right)} \int_0^\rho \sin^{n-i-1}(r) \ \cos^i(r) \, dr. \tag{4}$$

More generally, if M is an embedded submanifold of $H_n(\lambda)$, the n-dimensional model space of constant curvature λ then (c.f. [7])

$$\mathrm{Vol}_{H_n(\lambda)}\left(T(M, \rho)\right) = \sum_{j=0}^{n} \mathcal{L}_i^\lambda(M) \frac{2\pi^{(n-i)/2}}{\Gamma\left(\frac{n-i}{2}\right)} \int_0^\rho \frac{\sin^{n-i-1}(\sqrt{\lambda} r) \ \cos^i(\sqrt{\lambda} r)}{\lambda^{(n-i-1)/2}} \, dr. \tag{5}$$

The derivation of (2) and the verification of its precision depends on the spherical volume of tubes (4) as well as how M is embedded in $S(\mathbb{R}^n)$, specifically on its *critical radius*, see [11, 10]. Also, the proofs are very Gaussian in nature, that is, they shed no light on how to derive such approximations for, say a χ^2_k process, i.e. a sum of squares of k i.i.d. Gaussian processes.

An alternate approach which actually yields the same approximation as (2), the so-called Euler characteristic approach [1, 2, 3, 4, 16], "works" for the maximum of any smooth real-valued process. The quotation marks around the word "works" refers to the fact that rigorous results regarding the precision of the approximation in the general case have, up to now, evaded discovery. However, simulation studies show that the Euler characteristic (EC) approach is a practical, reasonably accurate approximation of the maximum of a smooth, real-valued stochastic process. The EC approach uses the following approximation, for large enough u,

$$\mathrm{P}\left[\sup_{p\in M} f(p) \geq u\right] \simeq \mathrm{E}\left[\chi\left(M \cap f^{-1}[u,+\infty)\right)\right] \tag{6}$$

which is based on the idea that, for large u, the sets $f^{-1}[u,+\infty)$ are "geodesically convex", combined with a Poisson clumping argument (c.f. [2]).

In the Gaussian case, Takemura and Kuriki [11] showed that the two approaches yield the same result, so that the accuracy of the Euler characteristic approach in the Gaussian case is the same as that of the volume of tubes approach.

When $y = (y_1, \dots, y_k)$ is a vector of i.i.d. centered unit-variance Gaussian processes on a manifold M, and, for $F \in C^2(\mathbb{R}^k)$ so that $f = F \circ y$ is a real-valued process on M (referred to here as a *Gaussian-related* process), the Euler characteristic approach leads to an expansion of the form (c.f. [12, 13])

$$\mathrm{P}\left[\sup_{p\in M} f(p) \geq u\right] \simeq \sum_{i=0}^{n} \mathcal{L}_i(M)\rho_{i,f}(u) \tag{7}$$

for functions $\rho_{i,f}(u)$ referred to as *Euler characteristic (EC) densities* [16] where $\mathcal{L}_i \overset{\Delta}{=} \mathcal{L}_i^0$ are the so-called *intrinsic volumes* of M [6, 8, 9].

As mentioned above, in their recent paper [11], Takemura and Kuriki show why, in the Gaussian case, the two approaches agree. They show that this equivalence is related to the classical Kinematic Fundamental Formulae of integral geometry [5, 6, 9] which state, that for embedded submanifolds M_1 and M_2 of $H_n(\lambda)$,

$$\int_{G_n(\lambda)} \mathcal{L}_k^\lambda(M_1 \cap gM_2)\, d\mu_n(g) = \sum_{j=0}^{n} c(n,j,k,\lambda)\mathcal{L}_{k+j}^\lambda(M_1)\mathcal{L}_{n-j}^\lambda(M_2) \tag{8}$$

where $G_n(\lambda)$ is the group of isometries of $H_n(\lambda)$ equipped with Haar measure μ_n and $c(n,j,k,\lambda)$ are universal constants. By the Chern-Gauss-Bonnet Theorem χ, the Euler characteristic can be written as a linear combination, dependent on λ, of the $\mathcal{L}_i^\lambda$ hence (8) implies

$$\int_{G_n(\lambda)} \chi(M_1 \cap gM_2)\, d\mu_n(g) = \sum_{j,k=0}^{n} \tilde{c}(n,j,k,\lambda)\mathcal{L}_j^\lambda(M_1)\mathcal{L}_l^\lambda(M_2) \tag{9}$$

for universal constants $\tilde{c}(n, j, k, \lambda)$ (different, but of course related to the $c(n, j, k, \lambda)$ above).

In their demonstration of this equivalence between the volume of tubes approach and the Euler characteristic approach, Takemura and Kuriki set M_1 to be M, a piecewise smooth manifold with boundary of $S(\mathbb{R}^n)$ and M_2 to be a spherical cap $C(r)$ of geodesic radius r. Hence, the $\mathcal{L}^1(M)$ in (2) above is related to the $\mathcal{L}_1(M_1)$ in (8) and the Q_i's are related to $\mathcal{L}_i^1(C(r))$. In particular, the rightmost terms of expressions (2) and (7) are also related to coefficients in an volume of tubes expansion, specifically of a tube around $C(r)$.

In this work, we describe this relation between EC densities and volume of tubes in detail for Gaussian related processes and present a Gaussian version of the Kinematic Fundamental Formulae. Specifically, the EC densities of a Gaussian related process can be shown to be coefficients in a power series expansion of the Gaussian measure of certain tubular neighbourhoods. Concretely,

$$\gamma_{\mathbb{R}^k}\left(T\left(F^{-1}[u,+\infty), r\right)\right) = \sum_{j=0}^{\infty} \frac{r^j}{j!}(2\pi)^{j/2}\rho_{j,f}(u), \tag{10}$$

where $\gamma_{\mathbb{R}^k}$ is the standard Gaussian measure on $\mathbb{R}^k$, i.e. the distribution of $\xi \sim N(0, I_{k\times k})$. This result is described in more detail in Section 2, after which we give some applications of the Gaussian KFF in Section 3, deriving the EC densities for Gaussian and non-central χ^2 fields.

We next use the Gaussian KFF to address a problem which has arisen in the context of brain imaging, called a conjunction analysis. A conjunction in a brain imaging experiment is said to occur if one region of the brain is active above a certain threshold under two or more different experimental conditions or for two or more subjects. In terms of processes, the set of "conjunction points" at the level u for two processes f_1 and f_2 is given by $f_1^{-1}[u,+\infty) \cap f_2^{-1}[u,+\infty)$, and similarly for higher order conjunctions. When the fields f_1 and f_2 are independent, a straightforward conditioning argument can be used to derive the Euler characteristic approximation of

$$\mathrm{P}\left[f_1^{-1}[u,+\infty) \cap f_2^{-1}[u,+\infty) \neq \emptyset\right],$$

the details of which can be found in [17].

This argument fails, however, when the fields are not independent. Consider the "correlated" conjunction given by the following processes

$$\begin{aligned} z_1 &= y_1 \\ z_2 &= \rho \cdot y_1 + \sqrt{1-\rho^2} \cdot y_2 \end{aligned}$$

for a vector $y = (y_1, y_2)$ of two i.i.d. centered, unit variance Gaussian processes on some manifold M. In this case, it is easy to show

$$z_1^{-1}[u,+\infty) \cap z_2^{-1}[u,+\infty) = y^{-1}\left(K(\rho, u)\right)$$

for some non-centered cone $K(\rho, u) \subset \mathbb{R}^2$. In Section 4, we expand the Gaussian volume of a tubular neighbourhood around $K(\rho, u)$ and use the Gaussian KFF to approximate

$$\mathrm{P}\left[z_1^{-1}[u, +\infty) \cap z_2^{-1}[u, +\infty) \neq \emptyset\right]. \tag{11}$$

In Section 5, we conclude with some simulation results to validate (11) which show that the approximation is quite accurate for a wide range of parameter spaces and values of ρ the correlation between the two fields.

2 Gaussian Kinematic Formula

In this section, we assume that we are given $y = (y_1, \ldots, y_k)$ i.i.d. zero-mean unit variance Gaussian processes on an n-dimensional manifold M as well as $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_k)$ a vector of k i.i.d. zero-mean, unit variance isotropic Gaussian process on $\mathbb{R}^n$. Further, we assume that each y_i and $\tilde{y}_i$ is suitably regular in the sense of [14], which holds if M is compact and each $y_i \in C^3(M), \mathrm{P}-$a.s. From y and $\tilde{y}$, we form new process by choosing $F \in C^2(\mathbb{R}^k)$ and defining

$$f = F \circ y \quad \text{and} \quad \tilde{f} = F \circ \tilde{y}.$$

One obvious choice of F is

$$F_{\mu,\nu}(x) = \langle x + \mu, \nu \rangle$$

for fixed vectors $\mu, \nu \in \mathbb{R}^k$ which results in another Gaussian process with mean $\langle \mu, \nu \rangle$ and variance $\|\nu\|^2$. Another example is

$$F_\xi(x) = \|x - \xi\|$$

for a fixed vector $\xi \in \mathbb{R}^k$ which results in a non-central χ^2_k field with non-centrality parameter $\|\xi\|$. Similarly, non-central F, Hotelling's T and other random fields can be defined.

A fundamental result of Worsley [16] for isotropic processes states that for compact C^2 domains T in $\mathbb{R}^n$

$$\mathrm{E}\left[\chi\left(T \cap \tilde{f}^{-1}[u, \infty)\right)\right] = \sum_{j=0}^{n} \mathcal{L}_j(T) \rho_{j,\tilde{f}}(u) \tag{12}$$

for functions $\rho_{j,\tilde{f}}(u)$ defined by

$$\rho_{j,\tilde{f}}(u) = \mathrm{E}\left[\det\left(\nabla^2 \tilde{f}(t)_{|j}\right) \middle| \frac{\partial \tilde{f}(t)}{\partial t_1} = 0, \ldots, \frac{\partial \tilde{f}(t)}{\partial t_j} = 0, \tilde{f}(t) \geq u\right] \tag{13}$$

for an arbitrary $t \in \mathbb{R}^n$ where

$$\nabla^2 \tilde{f}(t)_{|j} = \begin{pmatrix} \frac{\partial^2 f(t)}{\partial t_1 \partial t_1} & \cdots & \frac{\partial^2 f(t)}{\partial t_1 \partial t_j} \\ \frac{\partial^2 f(t)}{\partial t_2 \partial t_1} & \cdots & \frac{\partial^2 f(t)}{\partial t_2 \partial t_j} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(t)}{\partial t_j \partial t_1} & \cdots & \frac{\partial^2 f(t)}{\partial t_j \partial t_j} \end{pmatrix}.$$

From the above equation, it can be seen that the EC densities depend on the joint distribution of the field along with its first and second derivatives of the field f at a fixed point t, which means that calculating the EC densities, in general is a difficult task. However, the relation (10) does not depend on the distribution of the derivatives of the process, only on a set function on $\mathbb{R}^k$ so that in going from (13) to (10) all the spatial information regarding the derivative of the process has been "conditioned" out which means, in many cases a significant reduction in the dimensionality of the problem of calculating the EC densities.

We can write the expected Euler characteristic in (12) as

$$\mathrm{E}\left[\chi\left(T \cap \tilde{f}^{-1}[u,\infty)\right)\right] = \mathrm{E}\left[\chi\left(T \cap y^{-1}\left(\tilde{F}^{-1}[u,\infty)\right)\right)\right]$$

and the additivity of the Euler characteristic, i.e.

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

imply that the EC densities must satisfy

$$\rho_{j,\tilde{f}}(u) = \varphi_j\left(F^{-1}[u,+\infty)\right)$$

for some additive set functional φ_j on $\mathbb{R}^k$.

By direct, but delicate computation, Taylor [12, 13] showed that in fact for C^2 domains $D \subset \mathbb{R}^k$ (satisfying certain regularity conditions) $\varphi_j(\cdot)$ is equal to $(2\pi)^{-j/2}$ times the coefficient of $r^j/j!$ in a power series expansion for the standard Gaussian volume of a tubular neighbourhood of radius r around D which is denoted by $\mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(D)$. Note that the $\mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(\cdot)$'s are additive functionals by the same arguments that show the $\mathcal{L}_j(\cdot)$'s are additive. It was also shown that (12) holds for fields f on manifolds M as well, i.e. that

$$\mathrm{E}\left[\chi\left(M \cap f^{-1}[u,\infty)\right)\right] = \sum_{j=0}^{n} \mathcal{L}_j(M)\rho_{j,\tilde{f}}(u) \tag{14}$$

where $\mathcal{L}_j(M)$, $0 \leq j \leq n$ the intrinsic volumes of M, are computed with respect to the Riemannian metric induced by the process f, i.e. the metric g is defined by

$$g(X_p, Y_p) = \mathrm{E}\left[X_p y_i Y_p y_i\right].$$

Putting these facts together, we have the Gaussian Kinematic Formula

$$\mathrm{E}\left[\chi\left(M \cap y^{-1}D\right)\right] = \sum_{j=0}^{n} \mathcal{L}_j(M)(2\pi)^{-j/2}\mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(D). \tag{15}$$

We now apply the Gaussian KFF to specific processes, deriving the EC densities which gives the desired approximation (7).

3 Examples

3.1 The Gaussian case

Recall the definition of

$$F_{\mu,\nu}(x) = \langle x + \mu, \nu \rangle.$$

It is easy to see that

$$F_{\mu,\nu}^{-1}[u, +\infty) = \left\{ x \in \mathbb{R}^k : \frac{\langle x, \nu \rangle}{\|\nu\|} \geq \frac{u - \langle \mu, \nu \rangle}{\|\nu\|} \right\}$$

is a halfspace in $\mathbb{R}^k$ with measure

$$\gamma_{\mathbb{R}^k} \left(F_{\mu,\nu}^{-1}[u, +\infty) \right) = 1 - \Phi \left(\frac{u - \langle \mu, \nu \rangle}{\|\nu\|} \right)$$

with Φ the standard normal distribution function. As seen in Figure 3.1, $T \left(F_{\mu,\nu}^{-1}[u, +\infty), r \right)$ is also a halfspace in $\mathbb{R}^k$ with measure

$$\gamma_{\mathbb{R}^k} \left(F_{\mu,\nu}^{-1}[u, +\infty) \right) = 1 - \Phi \left(\frac{u - \langle \mu, \nu \rangle}{\|\nu\|} - r \right),$$

we can now expand the above as a power series in r to determine $\mathcal{M}_j^{\gamma_{\mathbb{R}^k}} \left(F_{\mu,\nu}^{-1}[u, +\infty) \right)$ and hence, the EC densities of $\tilde{f}_{\mu,\nu} = F_{\mu,\nu} \circ y$.

Without loss of generality, we now set $\mu = 0$ and $\nu = e_1$ the first coordinate vector so that the above becomes, exanding in r

$$\begin{aligned}
\gamma_{\mathbb{R}^k} \left(F_{0,e_1}^{-1}[u, +\infty) \right) &= 1 - \Phi\left(u - r\right) \\
&= \sum_{j=0}^{\infty} \frac{(-r)^j}{j!} \frac{d}{dx^j} \left(1 - \Phi\left(x\right)\right) \bigg|_{x=u} \\
&= 1 - \Phi(u) + \sum_{j=1}^{\infty} \frac{r^j}{j!} \frac{1}{\sqrt{2\pi}} \frac{d^{j-1}}{dx^{j-1}} \mathrm{e}^{-x^2/2} \bigg|_{x=u} \\
&= 1 - \Phi(u) + \sum_{j=1}^{\infty} \frac{(-r)^j}{j!} \frac{1}{\sqrt{2\pi}} (-1)^{j-1} H_{j-1}(u) \mathrm{e}^{-u^2/2} \\
&= 1 - \Phi(u) + \sum_{j=1}^{\infty} \frac{r^j}{j!} \frac{1}{\sqrt{2\pi}} H_{j-1}(u) \mathrm{e}^{-u^2/2}.
\end{aligned}$$

The Gaussian KFF (15) then implies that the EC densities of the field $\tilde{f}_{0,e_1} = F_{0,e_1} \circ \tilde{y}$, i.e. a centered, unit variance Gaussian process are given by, for $j \geq 1$

$$\rho_{j,\tilde{f}_{0,e_1}}(u) = \frac{1}{(2\pi)^{(j+1)/2}} H_{j-1}(u) \mathrm{e}^{-u^2/2}$$

a result originally derived in [1].

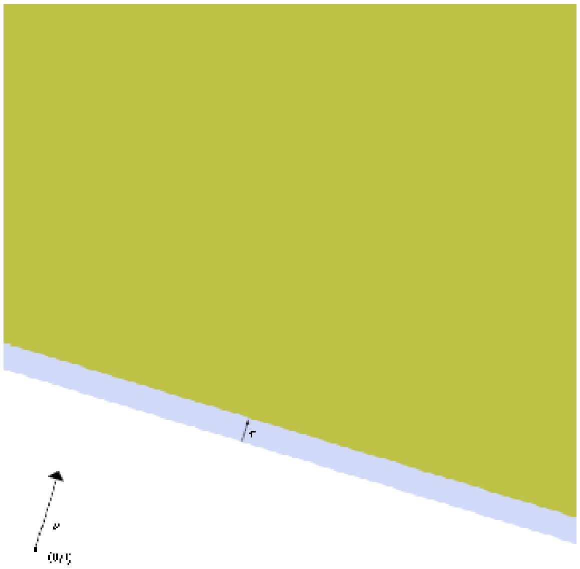

Figure 1: Tube of radius r around $F^{-1}_{\mu,\nu}[u,+\infty)$.

3.2 The χ^2 and non-central χ^2 case

In this subsection, we derive the EC densities of a χ^2_k random field, as defined in [1, 15]. Note that the EC densities of the χ^2_k were also derived in [15], but we rederive them here as a simple application of (10). The results in the non-central case are new and are just as easy an application of the Gaussian KFF.

Recall the definition of

$$F_\xi(x) = \|x-\xi\|.$$

Note that the field $\tilde{f}_\xi = F_\xi \circ \tilde{y}$ is the square root of a non-central χ^2_k process with non-centrality parameter $\|\xi\|$ on M and it is thus straightforward to obtain the EC densities of a non-central χ^2_k field, from those of its square root.

We first calculate the χ^2_k EC densities, from which the non-central ones can be calculated using the representation of its density as the sum of χ^2_k densities.

As in Figure 3.2, it is easy to see that

$$F_0^{-1}\big([u,+\infty)\big) = \overline{\mathbb{R}^k \backslash B_{\mathbb{R}^k}(0,u)},$$

where $B_{\mathbb{R}^k}(0,u)$ is the ball of radius u in $\mathbb{R}^k$ centered at the origin so that

$$T\left(F_0^{-1}\left([u,+\infty)\right),r\right) = \overline{\mathbb{R}^k \backslash B_{\mathbb{R}^k}(0,u-r)}.$$

This implies

$$\gamma_{\mathbb{R}^k}\big(T\left(F_0^{-1}\left([u,+\infty)\right),r\right)\big) = \mathrm{P}\left[\chi^2_k \geq (u-r)^2\right].$$

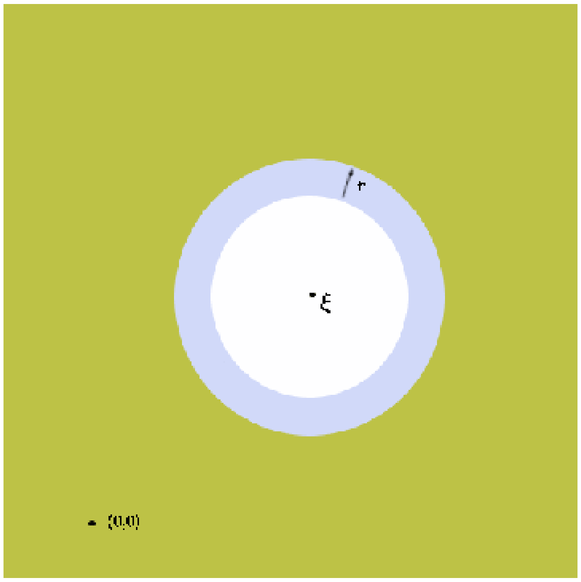

Figure 2: Tube of radii r around $C(v_1, v_2, w)$.

It remains to express the right hand side above as a Taylor series in r. The density f_k of the square root of a χ_k^2 random variable is

$$f_k(u) = \frac{1}{\Gamma(k/2)2^{(k-2)/2}} u^{k-1} \mathrm{e}^{-u^2/2},$$

so that

$$\gamma_{\mathbb{R}^k}\left(T\left(F_0^{-1}[u, +\infty)\right)\right) = \gamma_{\mathbb{R}^k}\left(F_0^{-1}[u, +\infty)\right) - \sum_{j=1}^{\infty} \frac{(-r)^j}{j!} \frac{d^{j-1} f_k(t)}{dt^{j-1}}\bigg|_{t=u}.$$

Direct calculations show that

$$\begin{aligned}
\frac{d^{j-1}f_k(t)}{dt^{j-1}} &= \frac{1}{\Gamma(k/2)2^{(k-2)/2}}\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i\frac{d^{j-1-i}t^{k-1}}{dt^{j-1-i}}H_i(t)\mathrm{e}^{-t^2/2} \\
&= \frac{1}{\Gamma(k/2)2^{(k-2)/2}}\sum_{i=0}^{j-1}1_{\{k\geq j-i\}}\binom{j-1}{i}(-1)^i\frac{(k-1)!}{(k+i-j)!}t^{k+i-j}H_i(t)\mathrm{e}^{-t^2/2} \\
&= \frac{t^{k-j}\mathrm{e}^{-t^2/2}}{\Gamma(k/2)2^{(k-2)/2}}\sum_{i=0}^{j-1}\sum_{l=0}^{\lfloor i/2\rfloor}1_{\{k\geq j-i\}}\binom{j-1}{i}(-1)^{i+l}\frac{(k-1)!}{(k+i-j)!}\frac{i!}{(i-2l)!l!2^l}t^{2i-2l} \\
&= \frac{t^{k-j}\mathrm{e}^{-t^2/2}}{\Gamma(k/2)2^{(k-2)/2}}\sum_{l=0}^{\lfloor\frac{j-1}{2}\rfloor}\sum_{i=2l}^{j-1}1_{\{k\geq j-i\}}\binom{j-1}{i}(-1)^{i+l}\frac{(k-1)!}{(k+i-j)!}\frac{i!}{(i-2l)!l!2^l}t^{2i-2l} \\
&= \frac{t^{k-j}\mathrm{e}^{-t^2/2}}{\Gamma(k/2)2^{(k-2)/2}}\sum_{l=0}^{\lfloor\frac{j-1}{2}\rfloor}\sum_{m=0}^{j-1-2l}1_{\{k\geq j-m-2l\}}\binom{k-1}{j-1-m-2l}\frac{(-1)^{m+l}(j-1)!}{m!l!2^l}t^{2m+2l}.
\end{aligned}$$

Combining the two equations gives

$$\begin{aligned}
&\gamma_{\mathbb{R}^k}\left(T\left(F_0^{-1}[u,+\infty)\right)\right) = \gamma_{\mathbb{R}^k}\left(F_0^{-1}[u,+\infty)\right) + \sum_{j=1}^{\infty}\frac{r^j}{j!}\frac{u^{k-j}\mathrm{e}^{-u^2/2}}{\Gamma(k/2)2^{(k-2)/2}}\times \\
&\qquad\left(\sum_{l=0}^{\lfloor\frac{j-1}{2}\rfloor}\sum_{m=0}^{j-1-2l}1_{\{k\geq j-m-2l\}}\binom{k-1}{j-1-m-2l}\frac{(-1)^{j-1+m+l}(j-1)!}{m!l!2^l}u^{2m+2l}\right).
\end{aligned}$$

We thus conclude, by (10) that for $j\geq 1$,

$$\begin{aligned}
&\rho_{j,\tilde{f}_0}(u) = \frac{u^{k-j}\mathrm{e}^{-u^2/2}}{(2\pi)^{j/2}\Gamma(k/2)2^{(k-2)/2}}\times \\
&\qquad\left(\sum_{l=0}^{\lfloor\frac{j-1}{2}\rfloor}\sum_{m=0}^{j-1-2l}1_{\{k\geq j-m-2l\}}\binom{k-1}{j-1-m-2l}\frac{(-1)^{j-1+m+l}(j-1)!}{m!l!2^l}u^{2m+2l}\right),
\end{aligned}$$

and thus the EC densities for $j\geq 1$ of the χ^2_k random field g are given by

$$\begin{aligned}
&\rho_{j,\tilde{f}_0^2}(u) = \frac{u^{(k-j)/2}\mathrm{e}^{-u/2}}{(2\pi)^{j/2}\Gamma(k/2)2^{(k-2)/2}}\times \\
&\qquad\left(\sum_{l=0}^{\lfloor\frac{j-1}{2}\rfloor}\sum_{m=0}^{j-1-2l}1_{\{k\geq j-m-2l\}}\binom{k-1}{j-1-m-2l}\frac{(-1)^{j-1+m+l}(j-1)!}{m!l!2^l}u^{m+l}\right),
\end{aligned}$$

which agrees with [15].

Recalling that the density $f_{\alpha,k}$ of the square root of a non-central χ^2_k random variable with non-centrality parameter α can be expressed as

$$f_{\alpha,k}(x) = \sum_{j=0}^{\infty}\mathrm{e}^{-\alpha^2/2}\frac{\alpha^j}{2^j j!}f_k(x),$$

where $f_k(x)$ is as above, we obtain the following relation for the non-central χ^2_k random fields $\tilde{f}_\xi = F_\xi \circ y$.

Lemma 3.1 *The EC densities of the non-central χ^2_k field with non-centrality parameter $\|\xi\|$ are*

$$\rho_{j,\tilde{f}_\xi}(u) = \sum_{i=0}^{\infty} \mathrm{e}^{-\|\xi\|^2/2} \frac{\|\xi\|^i}{2^i i!} \frac{u^{(k+2i-j)/2}\mathrm{e}^{-u/2}}{(2\pi)^{j/2}\Gamma((k+2i)/2)2^{(k+2i-2)/2}} \times$$
$$\left(\sum_{l=0}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{m=0}^{j-1-2l} 1_{\{k \geq j-m-2l-2i\}} \binom{k+2i-1}{j-1-m-2l} \frac{(-1)^{j-1+m+l}(j-1)!}{m!l!2^l} u^{m+l} \right).$$

4 Cones in $\mathbb{R}^2$ and the conjunction of two correlated Gaussian fields

In this section we study conjunctions of correlated Gaussian fields, which in terms of processes can be defined in terms of the minimum of two correlated Gaussian processes. Specifically, as in the introduction, given $y = (y_1, y_2)$ i.i.d. centered unit-variance Gaussian processes on some manifold M, we form two new Gaussian processes as follows

$$\begin{aligned} z_1 &= y_1 \\ z_2 &= \rho \cdot y_1 + \sqrt{1-\rho^2} \cdot y_2 \end{aligned}$$

and define $\tilde{z}_1$ and $\tilde{z}_2$, our isotropic versions of these processes as in Section 3. We define the conjunction of z_1 and z_2 at a point $p \in M$ by

$$z_1 \wedge z_2(p) \stackrel{\Delta}{=} \min(z_1(p), z_2(p)).$$

It is easy to see that

$$\begin{aligned} z_1 \wedge z_2^{-1}[u, +\infty) &= z_1^{-1}[u, +\infty) \cap z_2^{-1}[u, +\infty) \\ &= y^{-1}\left(K(u, \rho)\right) \end{aligned}$$

where $K(u, \rho)$ is a cone in $\mathbb{R}^2$ so that, to calculate the EC densities for this process, it suffices to calculate the expected Euler characteristic of $M \cap y^{-1}K$ for a general cone with arbitrary vertex in $\mathbb{R}^2$, which we proceed to do.

Define the cone

$$\begin{aligned} C(v_1, v_2, w) &= \{z \in \mathbb{R}^2 : z = w + a_1 v_1 + a_2 v_2, a_1, a_2 \geq 0\} \\ &= w + \{z \in T_w\mathbb{R}^2 : z = a_1 v_1 + a_2 v_2, a_1, a_2 \geq 0\} \end{aligned}$$

where $T_w\mathbb{R}^2$ is the tangent space to $\mathbb{R}^2$ at u. A sketch of the cone appears in Figure 4. We want to derive an expression for the following quantity

$$\mathrm{E}\big[\chi\big(M \cap y^{-1}C(v_1, v_2, w)\big)\big]$$

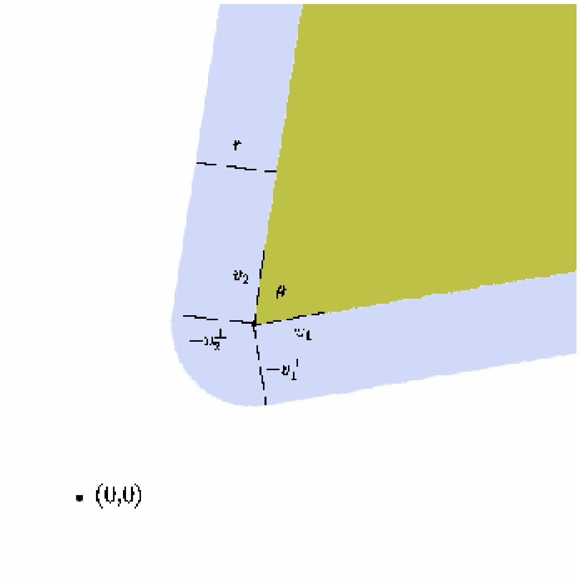

Figure 3: Tube of radius r around $C(v_1, v_2, w)$.

This random field arises in the study of conjunctions in brain imaging [17].

Associated to $C(v_1, v_2, w)$ are its normal cone

$$C^\perp(v_1, v_2, w) = \{z \in T_w\mathbb{R}^2 : \langle z, v_1\rangle < 0, \langle z, v_2\rangle < 0\},$$

with link

$$L\big(C^\perp(v_1, v_2, w)\big) = \{z \in S(T_w\mathbb{R}^2) : \langle z, v_1\rangle < 0, \langle z, v_2\rangle < 0\},$$

where $S(T_w\mathbb{R}^2)$ is the unit sphere in $T_w\mathbb{R}^2$.

In this case, the "domain" $C(v_1, v_2, w)$ is not smooth, however, for any $\delta > 0$, $T(C(v_1, v_2, w), \delta)$ is at least C^1 and the coefficients of

$$\gamma_{\mathbb{R}^2}\big(T(T(C(v_1, v_2, w), \delta), r)\big)$$

, as a Taylor series in r will, for small $\delta > 0$, be close to those of $\gamma_{\mathbb{R}^2}\big(T(C(v_1, v_2, w), r)\big)$, i.e. for small $\delta > 0$,

$$\begin{aligned}
\mathrm{E}\big[\chi\big(M \cap y^{-1}C(v_1, v_2, w)\big)\big] &\simeq \mathrm{E}\big[\chi\big(M \cap y^{-1}\big(T(C(v_1, v_2, w), \delta)\big)\big)\big] \\
&= \frac{1}{(2\pi)^{n/2}} \sum_{j=0}^{n} \mathcal{L}_j(M)\mathcal{M}_j^{\gamma_{\mathbb{R}^2}}\big(T(C(v_1, v_2, w), \delta)\big) \\
&\simeq \frac{1}{(2\pi)^{n/2}} \sum_{j=0}^{n} \mathcal{L}_j(M)\mathcal{M}_j^{\gamma_{\mathbb{R}^2}}\big(C(v_1, v_2, w)\big),
\end{aligned}$$

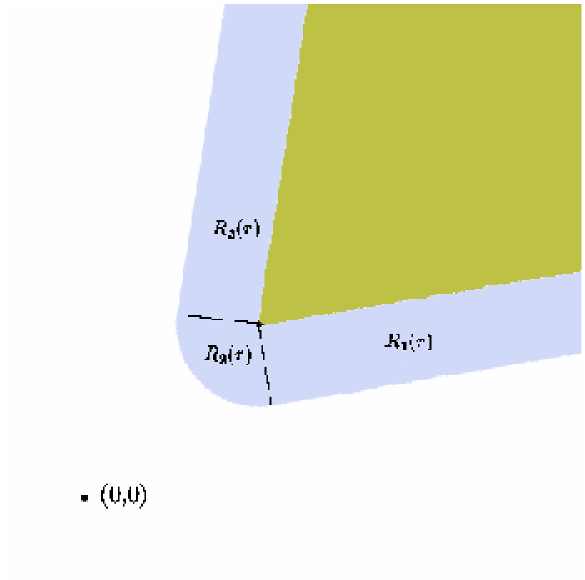

Figure 4: Regions of integration for power series expansion of $\gamma_{\mathbb{R}^2}\left(T\left(C(v_1, v_2, w)\right)\right)$.

where, for the cone $C(v_1, v_2, w)$, $\mathcal{M}_j^{\gamma_{\mathbb{R}^2}}(C(v_1, v_2, w))$ is defined as the coefficient of $r^j/j!$ in a Taylor series expansion of $\gamma_{\mathbb{R}^2}\big(T(D, r)\big)$. The expansion is split up into integrals over three regions, depicted in Figure 4.

Without loss of generality, we can choose a basis of $T_w\mathbb{R}^2$ so that the coefficients of v_1 are $(1, 0)$ and v_2 are $(\cos\theta, \sin\theta)$ where $\theta = \cos^{-1}(\langle v_1, v_2\rangle)$. We then set $v_1^\perp$ to be the unit vector orthogonal to v_1 such that $\langle v_1^\perp, v_2\rangle > 0$, i.e. the coefficients of $v_1^\perp$ with respect to our chosen basis are $(0, 1)$. The volume of the first region, $R_1(r)$ is thus

$$\begin{aligned}\gamma_{\mathbb{R}^2}\big(R_1(r)\big) &= \gamma_{\mathbb{R}^2}\big(\{z \in \mathbb{R}^2 : \langle v_1^\perp, z\rangle \in [\langle v_1^\perp, w\rangle - r, \langle v_1^\perp, w\rangle], \langle v_1, z\rangle \geq \langle v_1, w\rangle\}\big) \\ &= \frac{\big(1 - \Phi\big(\langle v_1, w\rangle\big)\big)}{\sqrt{2\pi}}\left(\sum_{j=1}^{\infty} \frac{r^j}{j!} H_{j-1}\big(\langle v_1^\perp, w\rangle\big) \mathrm{e}^{-\langle v_1^\perp, w\rangle^2/2}\right)\end{aligned}$$

By symmetry,

$$\begin{aligned}\gamma_{\mathbb{R}^2}\big(R_3(r)\big) &= \gamma_{\mathbb{R}^2}\big(\{z \in \mathbb{R}^2 : \langle v_2^\perp, z\rangle \in [\langle v_2^\perp, w\rangle - r, \langle v_2^\perp, w\rangle], \langle v_2, z\rangle \geq \langle v_2, w\rangle\}\big) \\ &= \frac{\big(1 - \Phi\big(\langle v_2, w\rangle\big)\big)}{\sqrt{2\pi}}\left(\sum_{j=1}^{\infty} \frac{r^j}{j!} H_{j-1}\big(\langle v_2^\perp, w\rangle\big) \mathrm{e}^{-\langle v_2^\perp, w\rangle^2/2}\right)\end{aligned}$$

Lastly, using polar coordinates

$$\begin{aligned}\gamma_{\mathbb{R}^2}\big(R_2(r)\big) &= \frac{1}{2\pi}\sum_{j=0}^{\infty}\int_{[0,r]\times L\left(C^{\perp}(v_1,v_2,w)\right)} \frac{t^{j+1}}{j!}\,\frac{d^j \mathrm{e}^{-\|z\|^2/2}}{d\nu^j}\bigg|_{z=w}\,dt\,d\nu \\ &= \frac{1}{2\pi}\sum_{j=2}^{\infty}\frac{r^j}{j!}(j-1)\int_{L\left(C^{\perp}(v_1,v_2,w)\right)} \frac{d^{j-2}\mathrm{e}^{-\|z\|^2/2}}{d\nu^{j-2}}\bigg|_{z=w}\,d\nu.\end{aligned}$$

Writing $\nu = -\sin\tilde{\theta}\cdot v_1 - \cos\tilde{\theta}\cdot v_1^{\perp}$, we have

$$\begin{aligned}\frac{d^{j-2}\mathrm{e}^{-\|z\|^2/2}}{d\nu^{j-2}}\bigg|_{z=w} &= (-1)^{j-2}\left(\sin\tilde{\theta}\frac{d}{dv_1}+\cos\tilde{\theta}\frac{d}{dv_1^{\perp}}\right)^{j-2}\mathrm{e}^{-\|z\|^2/2}\bigg|_{z=w} \\ &= \sum_{l=0}^{j-2}\binom{j-2}{l}\sin^{j-2-l}\tilde{\theta}\cos^{l}\tilde{\theta}\times \\ &\qquad H_{j-2-l}\big(\langle v_1,w\rangle\big)H_l\big(\langle v_1^{\perp},w\rangle\big)\mathrm{e}^{-\|w^2\|/2}.\end{aligned}$$

Noting that $\nu \in L\big(C^{\perp}(v_1,v_2,w)\big)$ if and only if $\tilde{\theta}\in(0,\pi-\theta)$ we see

$$\begin{aligned}\gamma_{\mathbb{R}^2}\big(R_2(r)\big) = \frac{1}{2\pi}\sum_{j=2}^{\infty}\frac{r^j}{j!}\sum_{l=0}^{j-2}(j-1)\binom{j-2}{l}H_{j-2-l}\big(\langle v_1,w\rangle\big)H_l\big(\langle v_1^{\perp},w\rangle\big)\mathrm{e}^{-\|w^2\|/2}\times \\ \int_0^{\pi-\theta}\sin^{j-2-l}\tilde{\theta}\cos^{l}\tilde{\theta}\,d\tilde{\theta}\end{aligned}$$

Straightforward calculations show

$$\begin{aligned}K_{j,l}(\theta) &\triangleq (j-1)\int_0^{\pi-\theta}\sin^{j-2-l}\tilde{\theta}\cos^{l}\tilde{\theta}\,d\tilde{\theta} \\ &= \begin{cases}\frac{j-1}{2}IB_{(j-1-l)/2,(l+1)/2}\big(\sqrt{\sin\theta}\big) & \text{if } \theta\ge\pi/2, \\ (-1)^l\frac{j-1}{2}\Big(B_{(j-1-l)/2,(l+1)/2}-IB_{(j-1-l)/2,(l+1)/2}\big(\sqrt{\sin\theta}\big)\Big)+ & \\ \qquad\frac{j-1}{2}B_{(j-1-l)/2,(l+1)/2} & \text{if } \theta\le\pi/2.\end{cases}\end{aligned}$$

where

$$IB_{\nu_1,\nu_2}(x) = \int_0^x t^{\nu_1-1}(1-t)^{\nu_2-1}\,dt$$

is the incomplete beta function and $B_{\nu_1,\nu_2} = IB_{\nu_1,\nu_2}(1)$ is the beta function.

Putting the above together, we have proved the following

Lemma 4.1 *The coefficient of $r^j/j!$ in the power series expansion of*

$$\gamma_{\mathbb{R}^2}\left(T\left(C(v_1,v_2,w),r\right)\right)$$

is given by

$$
\begin{aligned}
&\mathcal{M}_j^{\gamma_{\mathbb{R}^2}}\big(C(v_1, v_2, w)\big) \\
&= \begin{cases}
\gamma_{\mathbb{R}^2}\big(C(v_1, v_2, w)\big) & j = 0 \\
\frac{\big(1-\Phi(\langle v_1, w\rangle)\big)}{\sqrt{2\pi}} \mathrm{e}^{-\langle v_1^{\perp}, w\rangle^2/2} + \frac{\big(1-\Phi(\langle v_2, w\rangle)\big)}{\sqrt{2\pi}} \mathrm{e}^{-\langle v_2^{\perp}, w\rangle^2/2} & j = 1 \\
\frac{\big(1-\Phi(\langle v_1, w\rangle)\big)}{\sqrt{2\pi}} H_{j-1}\big(\langle v_1^{\perp}, w\rangle\big) \mathrm{e}^{-\langle v_1^{\perp}, w\rangle^2/2} + & \\
\qquad \frac{\big(1-\Phi(\langle v_2, w\rangle)\big)}{\sqrt{2\pi}} H_{j-1}\big(\langle v_2^{\perp}, w\rangle\big) \mathrm{e}^{-\langle v_2^{\perp}, w\rangle^2/2} + & \\
\qquad \frac{1}{2\pi} \sum_{l=0}^{j-2} \binom{j-2}{l} K_{j,l}(\theta) H_{j-2-l}\big(\langle v_1, w\rangle\big) H_l\big(\langle v_1^{\perp}, w\rangle\big) \mathrm{e}^{-\|w^2\|/2} & j \geq 2
\end{cases}
\end{aligned}
$$

It remains only to relate the above lemma to our original goal, that is the EC densities of the field $z_1 \wedge z_2$ which amounts to determining v_1, v_2 and w for the cone $K(u, \rho)$. We can take $v_1^{\perp} = (1, 0)$ and $v_2^{\perp} = (\rho, \sqrt{1-\rho^2})$ so that $v_1 = (0, 1)$ and $v_2 = (\sqrt{1-\rho^2}, -\rho)$ and $w = (u, u/\sqrt{1+\rho})$.

5 Simulation study

In order to validate the approximation (7) for the minimum of two Gaussian fields, a simulation study was done. We generated 2000 isotropic Gaussian processes on a $128 \times 128 \times 128$. The covariance of the processes was a Gaussian kernel with a 20 unit FWHM (Full Width at Half Maximum). To avoid wrap-around effects of the Fourier transform that was used to generate the data, we only considered cubes in the center of the region, from which we formed the empirical distribution of the maximum of $\tilde{z}_1 \wedge \tilde{z}_2$ for a range of values of ρ from -1 to 1 with steps of 0.1. From the empirical distribution functions, we found the $\alpha = 0.05$ quantile that would be used for a test of level 5%. In this study, we only considered cubical search regions in this study for simplicity, and we have plans to do a more elaborate study with varying shapes of search region.

Also plotted is a sample of the empirical distribution of the maximum of $\tilde{z}_1 \wedge \tilde{z}_2$ for a cube of side length 30 units to show the range over which the EC approximates (7). In this example, the cube side length is 30 units and, $\rho = 0.3$ though the other empirical distributions look quite similar.

6 Conclusion

In this work, we described a Gaussian version of the classical Kinematic Fundamental Formulae of integral geometry and used it to derive the EC densities of various random fields. In particular, we derived the EC densities for non-central χ^2 random processes as well as for the minimum of two correlated Gaussian processes, both of which have not previously been derived in the literature. To validate the usefulness of the approximation for the minimum of two Gaussian processes, the so-called “conjunction process”, a small simulation study was undertaken, the results of which show good agreement with the theoretical predictions of Section 4.

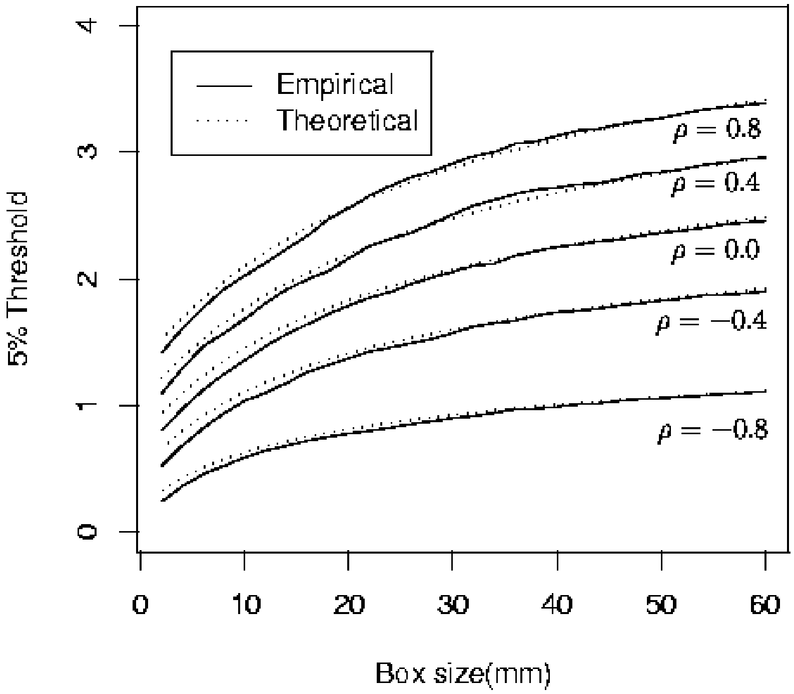

Figure 5: Threshold for $\alpha = 0.05$ test for cubes with side lengths between 2 and 60 units with FWHM=20 units and various values of correlation ρ.

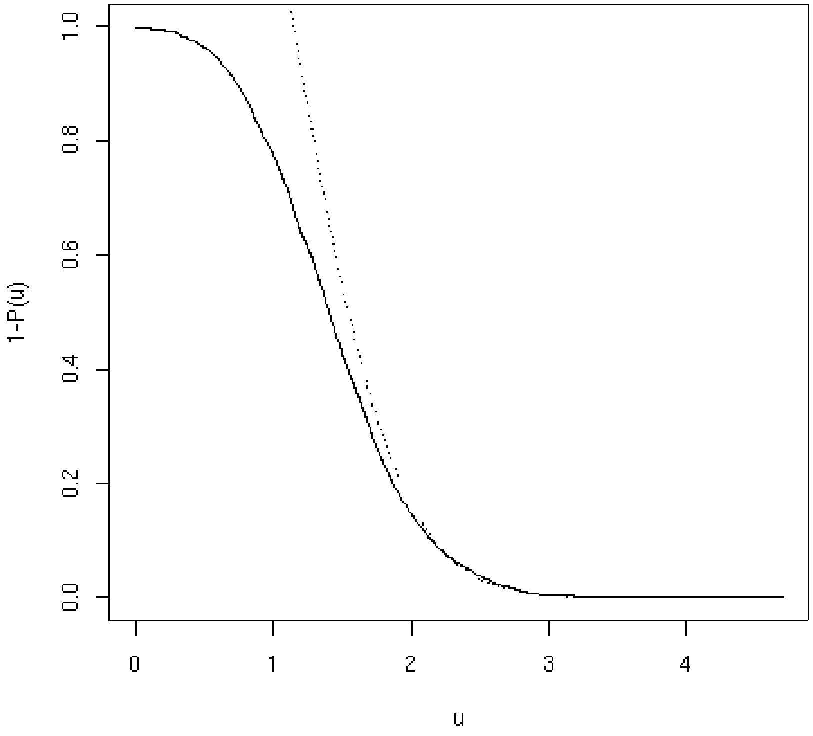

Figure 6: Empirical distribution and EC approximation for a cube with side length 30 units, with FWHM=20 units and $\rho = 0.3$.

References

[1] R. J. Adler. *The Geometry of Random Fields.* John Wiley & Sons, Chichester, 1981.

[2] R. J. Adler. On excursion sets, tube formulae, and maxima of random fields. *The Annals of Applied Probability*, 10(1):1–74, 2000.

[3] J. Cao and K. Worsley. The geometry of correlation fields with an application to functional connectivity of the brain. *Ann. Appl. Probab.*, 9(4):1021–1057, 1999.

[4] J. Cao and K. J. Worsley. The detection of local shape changes via the geometry of Hotelling's T^2 fields. *Ann. Statist.*, 27(3):925–942, 1999.

[5] S.-S. Chern. On the kinematic formula in integral geometry. *J. Math. Mech.*, 16:101–118, 1966.

[6] H. Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[7] A. Gray. *Tubes.* Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1990.

[8] D. A. Klain and G.-C. Rota. *Introduction to Geometric Probability.* Cambridge University Press, Cambridge, 1997.

[9] L. A. Santaló. *Integral Geometry and Geometric Probability.* Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1.

[10] A. Takemura and S. Kuriki. Tail probability via tube formula and Euler characteristic method when critical radius is zero. Technical Report CIRJE-F-59, CIRJE, Faculty of Economics, University of Tokyo, 1999.

[11] A. Takemura and S. Kuriki. Maximum of Gaussian field on piecewise smooth domain: Equivalence of tube method and Euler characteristic method. *Ann. of Appl. Prob.*, 12(2):768–796, 2002.

[12] J. Taylor. *Euler characteristics for Gaussian fields on manifolds.* PhD thesis, McGill University, 2001.

[13] J. E. Taylor. A Gaussian kinematic formula. Technical Report 2002-2, Department of Statistics, Stanford University, 2002.

[14] J. E. Taylor and R. J. Adler. Euler characteristics for Gaussian fields on manifolds. *Annals of Probability*, 2000. Submitted.

[15] K. J. Worsley. Local maxima and the expected Euler characteristic of excursion sets of χ^2, F and t fields. *Adv. in Appl. Probab.*, 26(1):13–42, 1994.

[16] K. J. Worsley. Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics. *Adv. in Appl. Probab.*, 27(4):943–959, 1995.

[17] K. J. Worsley and K. J. Friston. A test for a conjunction. *Statist. Probab. Lett.*, 47(2):135–140, 2000.