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Persi Diaconis; Charles Stein

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SOME TAUBERIAN THEOREMS RELATED TO COIN TOSSING

BY PERSI DIACONIS AND CHARLES STEIN

Stanford University

Let A be a subset of the integers and let S_n be the number of heads in n tosses of a p coin. If $\lim_{n\to\infty} P(S_n \in A)$ exists for some p then the limit exists for all p and does not depend on p. The relation of the limit to the density of A and to a similar Poisson limit is also given.

1. Introduction and statement of results. We consider the following problem. Let A be a subset of the integers (such as the even numbers). Let S_n be the number of heads in n tosses of a coin. Does $\lim_{n\to\infty} P(S_n \in A)$ exist, and how is it related to the set A?

A subset A of the set N of all nonnegative integers is said to have Euler density l (with parameter $p \in (0, 1)$) if

$$\lim_{n\to\infty} \textstyle\sum_{i\in A} b(i, n, p) = l \tag{1.1}$$

where

$$\begin{aligned} b(i, n, p) &= \tbinom{n}{i} p^i (1-p)^{n-i} && \text{for} \quad i \in \{0, \cdots, n\} \\ &= 0 && \text{otherwise.} \end{aligned}$$

We shall also say that A has E_p density l if (1.1) holds. This notion was introduced by Euler in order to manipulate divergent series. Modern references are Hardy (1949) and Peyerimhoff (1969). The principal result of this paper is that the existence and value of E_p density does not depend on the value of p. In greater detail, we have

THEOREM 1. *For any $A \subset N$ and $p \in (0, 1)$ the following assertions are equivalent:*

$$A \quad \textit{has} \quad E_p \quad \textit{density} \quad l\,, \tag{1.2}$$

$$\lim_{\lambda\to\infty} e^{-\lambda} \textstyle\sum_{i\in A} \dfrac{\lambda^i}{i!} = l\,, \tag{1.3}$$

$$\textit{for all} \quad \varepsilon > 0\,, \qquad \lim_{n\to\infty} \frac{1}{\varepsilon n^{\frac12}} \textstyle\sum_{i\in A;\, n\le i< n+\varepsilon n^{\frac12}} 1 = l\,. \tag{1.4}$$

This was conjectured independently by Erdös and Gleason, at least in part. The proof is an application of Wiener's Tauberian theorem after an appropriate variance-stabilizing transformation has been made and the normal approximation to the binomial used. Variance-stabilizing transformations are described in Anscombe (1948).

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An example will be useful in comparing rates of convergence of different densities.

EXAMPLE 2. If A is the set of multiples of an integer a, then

$$\left|\sum_{i \in A} b(i, n, p) - \frac{1}{a}\right| \leqq e^{-8np(1-p)/a^2}$$

This is proved below. Thus A has E_p density $1/a$ for any fixed p, $0 < p < 1$.

Consider the problem of the random division of a set of n counters into two piles. Is the uniform distribution on $\{0, 1, 2, \cdots, n\}$ a reasonable model for the number of objects in one of the piles? Laplace, in a controversy reported in Todhunter (1965, pages 200, 465), argued that the binomial model was more appropriate in determining if the number of counters in one pile was odd or even. Gardner (1973) discusses the "random" division of a pile of sticks in connection with the randomization mechanism of the I Ching. For mathematical convenience the uniform model is used in determining if the remainder in one of the piles is divisible by 4.

Theorem 3 below states that if n is reasonably large the uniform and binomial models lead to approximately the same answer.

Often the most readily available information about a set of integers is that it has Cesaro density (C_1 density) with a given rate of convergence. The next theorem asserts that if this rate is better than $1/n^{\frac{1}{2}}$ the set has E_p density. A positive, measurable, real-valued function L is *slowly varying at infinity* if for any $a > 0$, $\lim_{x \to \infty} L(ax)/L(x) = 1$. A measurable real-valued function is said to *vary regularly at infinity* with exponent ρ, $-\infty < \rho < \infty$, if $f(x) = x^\rho L(x)$ for $0 \leqq x < \infty$. Seneta (1976) contains the basic facts about regular variation. We can now state:

THEOREM 3. *If $A \subset N$ has E_p density l then A has C_1 density l. Conversely, if*

$$\left|\left(\frac{1}{n} \sum_{i \in A; i \leqq n} 1\right) - l\right| \leqq g(n) \tag{1.5}$$

where $g(x)$ varies regularly at infinity with exponent ρ, $-1 < \rho \leqq -\frac{1}{2}$, then

$$\left|\sum_{i \in A} b(i, n, p) - l\right| \leqq kg(n)n^{\frac{1}{2}} \tag{1.6}$$

for some constant k.

The set A, which does not include 0, includes 1, 2, 3, does not include 4 through 7, includes the next five integers, and so on, is easily seen to have C_1 density $\frac{1}{2}$ but is not $E_{\frac{1}{2}}$ summable to any limit. Similarly, it is easy to construct examples of sets A with arbitrarily slow rate of convergence in (1.5) which have E_p density. Example 2 shows that the information provided by (1.6) can be far from best possible. Here is an example where the rate of convergence given by Theorem 3 is the best rate available. An integer is *square-free* if it has no squared prime factors. Let $Q \subset N$ be the set of square-free numbers. Using a known result due to Walfisz along with Theorem 3 yields

COROLLARY 4.

$$\left|\textstyle\sum_{i\in Q} b(i, n, p) - \frac{6}{\pi^2}\right| \leqq E(n)$$

where

$$E(n) \leqq C_1 \exp\{-C_2 \log^{\frac{3}{5}} n(\log\log n)^{-\frac{1}{5}}\}$$

for C_1, C_2 *constants. If the* Riemann *hypothesis is true, then* $E(n) \leqq C_3 n^{-\frac{1}{10}+\varepsilon}$ *for any* $\varepsilon > 0$.

Returning to probabilistic language, Corollary 4 suggests a probabilistic way to determine π by flipping a coin n times and determining if the number of heads is square-free or not.

2. Proof of theorems.

PROOF OF THEOREM 1. That (1.2) implies (1.3) is Theorem 128 in Hardy (1949). The main steps of the remaining parts of the proof will be stated as a sequence of auxiliary lemmas. Letting $A \subset N$ be a fixed set throughout the proof, define the real-valued step function:

$$\begin{aligned} (2.1) \qquad f(x) &= 1 \quad \text{if} \quad 2(2i)^{\frac{1}{2}} \leqq x < 2(2(i+1))^{\frac{1}{2}} \quad \text{for some} \quad i \in A \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

LEMMA 5. *A has* E_p *density* l *if and only if*

$$(2.2) \qquad \lim_{t\to\infty} (2\pi\sigma^2)^{-\frac{1}{2}} \textstyle\int_{-\infty}^{\infty} f(x+t)e^{-x^2/2\sigma^2}\,dx = l \qquad \text{with} \quad \sigma^2 = 2(1-p)\,.$$

LEMMA 6. (1.3) *holds if and only if* (2.2) *holds with* $\sigma^2 = 2^{\frac{1}{2}}$.

LEMMA 7. (1.4) *holds if and only if for any* $\varepsilon > 0$

$$(2.3) \qquad \lim_{t\to\infty} \frac{1}{\varepsilon} \textstyle\int_0^{\varepsilon} f(x+t)\,dx = l\,.$$

Since the Fourier transform of the function

$$(2\pi\sigma^2)^{-\frac{1}{2}}e^{-x^2/2\sigma^2}$$

does not vanish, Wiener's Tauberian theorem (see, for example, Hardy (1949) Theorem 220) implies that if the limit in (2.2) exists for any $\sigma^2 > 0$ it exists for all $\sigma^2 > 0$. Thus (1.2) is equivalent to (1.3). Further, Wiener's Tauberian theorem implies that if the limit (2.2) exists then the limit (2.3) exists. Thus (1.2) or (1.3) imply (1.4). That (1.4) implies (1.2) requires a separate argument.

In what follows, k denotes an unspecified constant which need not be the same from equation to equation.

PROOF OF LEMMA 5. Using bounds for the normal approximation to the binomial measure, as given in Feller (1968, Chapter 7, formulas 3.9 and 3.10) we see that

$$(2.4) \qquad \textstyle\sum_{i\in A} \binom{n}{i} p^i(1-p)^{n-i} = \sum_{i\in A} (2\pi np(1-p))^{-\frac{1}{2}} \exp\left\{\frac{-(i-np)^2}{2np(1-p)}\right\} + R_{n,i}$$

where $R_{n,i}$ satisfies

$$\sum_{i=0}^{\infty} |R_{n,i}| \leqq \frac{k}{n^{\frac{1}{2}}}\,. \tag{2.5}$$

Next we compute, writing $\sigma^2 = 2(1 - p)$,

$$\begin{aligned}(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x + 2(2np)^{\frac{1}{2}})e^{-x^2/2\sigma^2}\, dx \\ = \sum_{i \in A} (2\pi\sigma^2)^{-\frac{1}{2}} \int_{2(2i)^{\frac{1}{2}}-2(2np)^{\frac{1}{2}}}^{2(2(i+1))^{\frac{1}{2}}-2(2np)^{\frac{1}{2}}} e^{-x^2/2\sigma^2}\, dx \\ = \sum_{i \in A} \frac{e^{-2((2i)^{\frac{1}{2}}-(2np)^{\frac{1}{2}})^2/\sigma^2}}{(\pi i\sigma^2)^{\frac{1}{2}}} + R'_{n,i}\end{aligned} \tag{2.6}$$

where $R'_{n,i}$ satisfies a condition analogous to (2.5). In comparing the sums in (2.4) and (2.6) we are free to only consider i satisfying

$$i \in S_n \quad \text{where} \quad S_n = \{i\colon |i - np| < n^{\frac{1}{2}} \log^2 n\} \tag{2.7}$$

since well-known bounds on the tails of these sums (Feller (1968), page 151) show they are negligible for large n. Thus, the difference between (2.4) and (2.6) is bounded by

$$\sum_{i \in S_n} \frac{e^{-(i-np)^2/2np(1-p)}}{(2\pi np(1 - p))^{\frac{1}{2}}} \left\{1 - \left(\frac{np}{i}\right)^{\frac{1}{2}} e^{f(i,n,p)}\right\} + o(1)\,, \tag{2.8}$$

where

$$\begin{aligned} f(i, n, p) &= \left\{\frac{(i - np)^2}{2np(1 - p)} - \frac{1}{(1 - p)}((2i)^{\frac{1}{2}} - (2np)^{\frac{1}{2}})^2\right\} \\ &= \frac{((2i)^{\frac{1}{2}} - (2np)^{\frac{1}{2}})^3}{8np(1 - p)} \{(2i)^{\frac{1}{2}} + 3(2np)^{\frac{1}{2}}\}\,. \end{aligned}$$

A straightforward argument shows that, for $i \in S_n$,

$$f(i, n, p) = O\left(\frac{(\log n)^8}{n^{\frac{1}{2}}}\right), \tag{2.9}$$

while clearly, for $i \in S_n$,

$$\left(\frac{np}{i}\right)^{\frac{1}{2}} = 1 + O\left(\frac{(\log n)^2}{n^{\frac{1}{2}}}\right).$$

Using this and (2.9) in (2.8) shows that the sum in (2.8) goes to zero as n goes to ∞. It remains to relate the integral on the left of expression (2.6) to the integral (2.2). Toward this end we calculate for u and t real numbers,

$$\begin{aligned}|(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \{f(x + t) - f(x + u)\}e^{-x^2/2\sigma^2}\, dx| \\ \leqq (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |e^{-(x-t)^2/2\sigma^2} - e^{-(x-u)^2/2\sigma^2}|\, dx \\ = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-|u-t|/2^{\frac{1}{2}}}^{|u-t|/2^{\frac{1}{2}}} e^{-x^2/\sigma^2}\, dx \\ \leqq (\pi\sigma^2)^{-\frac{1}{2}}|u - t|\,.\end{aligned} \tag{2.10}$$

For given t define $n(t) = [t^2/8p]$ where $[x]$ denotes the greatest integer less than or equal to x. Then clearly $2(2pn(t))^{\frac{1}{2}} = t + O(1/t)$ as $t \to \infty$. This and (2.10) show that

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} f(x+t)e^{-x^2/2}\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x+2(2pn)^{\frac{1}{2}})e^{-x^2/2}\,dx$$
$$= \lim_{n\to\infty} \sum_{i\in A} b(i, n, p)\,.$$

This completes the proof of Lemma 5. □

PROOF OF LEMMA 6. The normal approximation to the Poisson measure as in Feller (1968, page 194) or Hardy (1949, Theorem 137) implies that

$$e^{-\lambda} \sum_{i\in A} \frac{\lambda^i}{i!} = \sum_{i\in A} \frac{e^{-(i-\lambda)^2/2\lambda}}{(2\pi\lambda)^{\frac{1}{2}}} + R_{i,\lambda}$$

where $\sum_{i=0}^{\infty} |R_{i,\lambda}| < k/\lambda^{\frac{1}{2}}$. From here, the proof of Lemma 5 holds essentially word for word. □

PROOF OF LEMMA 7. We easily compute that

$$\frac{1}{\varepsilon} \int_0^{\varepsilon} f(x+t)\,dx = \frac{1}{t}\{\sum_{t<2(2i)^{\frac{1}{2}}<t+\varepsilon;\, i\in A} (2(2(i+1))^{\frac{1}{2}} - 2(2i)^{\frac{1}{2}})\} + O\left(\frac{1}{t}\right)$$
$$= \frac{4}{\varepsilon t} \sum_{t^2/8<i<(t+\varepsilon)^2/8;\, i\in A} 1 + O\left(\frac{1}{t}\right)$$
$$= \frac{4}{\varepsilon t} \sum_{t^2/8<i<t^2/8+\varepsilon t/4;\, i\in A} 1 + O\left(\frac{1}{t}\right)$$
$$= \frac{1}{\varepsilon(x/2)^{\frac{1}{2}}} \sum_{x<i<x+\varepsilon(x/2)^{\frac{1}{2}};\, i\in A} 1 + O\left(\frac{1}{x^{\frac{1}{2}}}\right)$$

where $x = t^2/8$. The last sum is (1.4) with ε replaced by $\varepsilon/2^{\frac{1}{2}}$. This completes the proof of Lemma 7. □

It only remains to show that (1.4) implies (1.2). Let

$$b_i = 1 \quad \text{if} \quad i \in A$$
$$= 0 \quad \text{otherwise.}$$

Let $a_i = b_i - l$. Condition (1.4) becomes

(2.11) for any $\varepsilon > 0$, $\delta > 0$, there is an N so that for $n > N$,
$|\sum_{n\leq i\leq n+\varepsilon n^{\frac{1}{2}}} a_i| \leq \delta n^{\frac{1}{2}}$.

We first show that if (2.11) holds, then it holds uniformly in ε. Specifically,

(2.12) for any positive real numbers $\delta < a < b$, . there is an N so that for $n > N$, $|\sum_{n\leq i\leq n+sn^{\frac{1}{2}}} a_i| \leq \delta n^{\frac{1}{2}}$ for $s \in [a, b]$.

To prove (2.12), find N so large that $n > N$ implies $b/n^{\frac{1}{2}} < 1$ and

(2.13) $$|\sum_{n\leq i\leq n+rn^{\frac{1}{2}}} a_i| \leq tn^{\frac{1}{2}}$$

with $r = \delta/8$, $t = \min(\delta/8, \delta^2/16b)$. Then, for $n > N$, let $x_0 = n$. $x_1 = n + rn^{\frac{1}{2}}$, and inductively, $x_{i+1} = x_i + r(x_i)^{\frac{1}{2}}$. Let I be the index such that $x_I \leq n + sn^{\frac{1}{2}} < x_{I+1}$.

Thus $x_{I+1} - x_I \leqq 2rn^{\frac{1}{2}}$ and

$$\begin{aligned} |\textstyle\sum_{n \leqq i \leqq n+sn^{\frac{1}{2}}} a_i| &\leqq \textstyle\sum_{i=0}^{I} |\sum_{x_i \leqq j < x_{i+1}} a_j| + 2rn^{\frac{1}{2}} \\ &\leqq \textstyle\sum_{i=0}^{I} t(x_i)^{\frac{1}{2}} + 2rn^{\frac{1}{2}} = \frac{t}{r}(x_{I+1} - x_0) + 2rn^{\frac{1}{2}} \\ &\leqq \frac{ts}{r} n^{\frac{1}{2}} + 2tn^{\frac{1}{2}} + 2rn^{\frac{1}{2}} \\ &\leqq \delta n^{\frac{1}{2}} . \end{aligned}$$

This completes the proof of (2.12). □

We must show that

$$\left|\frac{1}{2^n} \textstyle\sum_{i=0}^{n} \binom{n}{i} a_i\right| < \delta . \tag{2.14}$$

Using the boundedness of the sequence a_i and the central limit theorem (Feller (1968), Chapter 7), we first choose numbers N_1, w and z, such that for $n > N_1$,

$$\left|\frac{1}{2^n} \textstyle\sum_{i=0}^{n} \binom{n}{i} a_i\right| \leqq \frac{\delta}{10} + \left|\frac{1}{2^n} \textstyle\sum_{n/2 - wn^{\frac{1}{2}} \leqq i \leqq n/2 + zn^{\frac{1}{2}}} \binom{n}{i} a_i\right| . \tag{2.15}$$

Let $N_2 > N_1$ be so large that for $n > N_2$,

$$|\textstyle\sum_{i \in S(s)} a_i| \leqq \frac{\delta}{10} n^{\frac{1}{2}} \tag{2.16}$$

uniformly for $0 < s < z + w = b$ where $S(s) = \{i : n/2 - wn^{\frac{1}{2}} \leqq i \leqq n/2 - wn^{\frac{1}{2}} + sn^{\frac{1}{2}}\}$. Summation by parts now shows that the sum on the right side of (2.15) can be written as

$$\frac{1}{2^n} \textstyle\sum_{i \in S(b)} \left(\binom{n}{i} - \binom{n}{i+1}\right) A(i) + g(n) \tag{2.17}$$

where

$$A(i) = \textstyle\sum_{j = n/2 - wn^{\frac{1}{2}}}^{i} a_i , \qquad |g(n)| \leqq \frac{\delta}{10} .$$

Using (2.16), the sum in (2.17) is bounded by

$$\frac{n^{\frac{1}{2}}}{2^n} \frac{\delta}{10} \textstyle\sum_{i \in S(b)} \left|\binom{n}{i} + \binom{n}{i+1}\right| + \frac{\delta}{10} . \tag{2.18}$$

We also clearly have

$$\textstyle\sum_{i \in S(b)} \left|\binom{n}{i} - \binom{n}{i+1}\right| \leqq \sum_{i=0}^{n-1} \left|\binom{n}{i} - \binom{n}{i+1}\right| = 2\left\{\binom{n}{[n/2]} - 1\right\} \leqq \frac{k2^n}{n^{\frac{1}{2}}} .$$

Using this in (2.18) completes the proof of (2.14) and Theorem 1.

PROOF OF EXAMPLE 2. If $w = e^{2\pi i/a}$ is a primitive ath root of unity, it is elementary that for integers h,

$$\begin{aligned} \textstyle\sum_{j=0}^{a-1} w^{jh} &= a \quad \text{if } a \text{ divides } h \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

From this we have an identity of C. Ramus (Knuth (1973), page 70):

$$\frac{1}{a} \textstyle\sum_{j=0}^{a-1} (w^j p + (1-p))^n = \sum_{j \geqq 0} \binom{n}{aj} p^j (1-p)^{n-j} \tag{2.19}$$

(using the convention that $\binom{b}{c} = 0$ for integers $c > b$). The left-hand side of (2.19) is $1/a + E(n, p, a)$ where the error term

$$E(n, p, a) = \frac{1}{a} \textstyle\sum_{j=1}^{a-1} (pe^{2\pi ij/a} + (1+p))^n .$$

Writing $\phi = 2\pi j/a$, define R and θ by $((1-p) + pe^{i\phi}) = Re^{i\theta}$. Thus $R^2 = (1 - 4p(1-p)\sin^2(\phi/2))$. A Taylor expansion shows that $\log R < 2p(1-p)\sin^2(\phi/2)$. Use of $\sin x \geqq (2/\pi)x$ for $0 \leqq x \leqq \pi/2$ yields:

$$\begin{aligned} |(1-p) + pe^{2\pi ij/a}|^n &= |(1-p) + pe^{2\pi i(a-j)/a}|^n = |R|^n = e^{n \log R} \\ &\leqq e^{-np(1-p)\sin^2(\phi/2)} \leqq e^{-8np(1-p)j^2/a} . \end{aligned}$$

The required upper bound for $E(n, p, a)$ follows by replacing j by 1.

Proof of Theorem 3. If $A \subset N$ has E_p density l, then it is well known that $\lim_{x \to 1^-} (1-x) \sum_{i \in A} x^i = l$. Indeed, (1.2) implies (1.3) and Peyerimhoff (1969), Theorem III, page 17 shows that (1.3) implies the indicator function of A has Able density. The Hardy–Littlewood Tauberian theorem (Hardy (1949), Theorem 95) implies that A has C_1 density l. For the converse, let

$$\begin{aligned} a_i &= 1 \quad i \in A \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Assume that $\sum_{i=0}^n a_i = ln + O(f(n))$ where $f(n)$ varies regularly at infinity with exponent ρ, $0 < \rho \leqq \frac{1}{2}$. Throughout the proof let p, $0 < p < 1$, be fixed and let $q = 1 - p$. We must show that

$$\textstyle\sum_{i=0}^n a_i p^i q^{n-i} \binom{n}{i} = l + O\left(\dfrac{f(n)}{n^{\frac{1}{2}}}\right) . \tag{2.20}$$

Write the left side of (2.20) as

$$\textstyle\sum_{i=0}^n a_i p^i q^{n-i} \binom{n}{i} = q^n \sum_{i=0}^n a_i \binom{n}{i} r^i$$

with $r = p/q$. It is convenient to deal with the sum in the second form. Summation by parts gives

$$\textstyle\sum_{i=0}^n a_i \binom{n}{i} r^i = A(n) + \sum_{j=0}^{n-1} A(j)\Delta(j) \tag{2.21}$$

where $A(j) = \sum_{i=0}^j a_i$ and

$$\Delta(j) = \binom{n}{j} r^j - \binom{n}{j+1} r^{j+1} = \binom{n}{j} r^j \left\{ \frac{(r+1)j + 1 - rn}{j+1} \right\} . \tag{2.22}$$

Using the hypothesis, (2.21) becomes

$$\begin{aligned} &= O(n) + l \textstyle\sum_{j=1}^{n-1} j\Delta(j) + \sum_{j=1}^{n-1} O(f(j))\Delta(j) \\ &= O(n) + \frac{l}{q^n} + O(\textstyle\sum_{j=1}^{n-1} |\Delta(j)| f(j)) . \end{aligned} \tag{2.23}$$

We now bound the sum on the right side of (2.23).

$$\sum_{j=1}^{n-1} |\Delta(j)| f(j) \leqq \max_{1 \leqq j \leqq n} f(j) \{ \sum_{i=1}^{n-1} |\Delta(i)| \} \tag{2.24}$$

$$\leqq k f(n) \sum_{i=1}^{n-1} |\Delta(i)| \leqq \frac{k' f(n)}{q^n n^{\frac{1}{2}}} \,.$$

In the next to last inequality we have used the fact that $f(n) \sim \sup_{1 \leqq x \leqq n} f(x)$ as on pages 19–20 of Seneta (1976). From (2.22) the $\Delta(j)$ are of constant sign for $0 < j \leqq np - 1$ and for $np - q < j \leqq n$. Thus the last inequality in (2.24) follows from the standard bound for the maximal term of the binomial distribution (Feller (1968), page 151). Using (2.24) in (2.23) completes the proof of Theorem 3. □

PROOF OF COROLLARY 4. Let

$$Q(x) = \sum_{i \in x; i \in Q} 1$$

and

$$E(x) = Q(x) - x \frac{6}{\pi^2} \,.$$

Walfisz (1962) proved that

$$E(x) = O(x^{\frac{1}{2}} \exp\{-c \log^{\frac{3}{5}} x (\log \log x)^{-\frac{1}{5}}\})$$

while Vaidya (1966) proved that if the Riemann hypothesis is true, then

$$E(x) = O(x^{\frac{2}{5}+\varepsilon}) \,.$$

The stated results now follow from Theorem 3 by noting in both cases the bound for $E(x)$ varies regularly at ∞ with suitable exponent. □

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305