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BUFFON'S PROBLEM WITH A LONG NEEDLE

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Abstract

A needle of length l dropped at random on a grid of parallel lines of distance d apart can have multiple intersections if $l > d$. The distribution of the number of intersections and approximate moments for large l are derived. The distribution is shown to converge weakly to an arc sine law as $l/d \to \infty$.

BUFFON'S PROBLEM; GEOMETRICAL PROBABILITY; METHOD OF MOMENTS

1. Introduction

In the classical formulation of the Buffon needle problem ([1], p. 70) a needle of length l is thrown at random onto a plane ruled by parallel lines distance d apart, and one asks for the probability of an intersection. In case $l > d$ there can be several intersections. The purpose of this note is to discuss the probability, and approximations to the moments, of the number of crossings.

This problem was suggested by Herbert Solomon from consideration of a problem of detection deployment. In Solomon's problem, the planar grid is a grid of detection lines (e.g., a light or laser shining onto a photoelectric cell) and the needle might be a stream of polluting material laid down at random by a ship or a plane. The probability of the number of crossings is as given by Kendall and Moran ([3], p. 73). Other authors who have derived partial versions of Theorem 1 include Gridgeman [2] and Uspensky ([5], p. 258). The new results in this note are the simple expression (2.2) for the distribution function, the approximate moments, valid for large values of the ratio l/d given in Theorem 2 and the limit theorem for the density, Theorem 3.

2. Properties of the distribution of crossings

A needle of length l is thrown onto a plane ruled by parallel lines distance d apart with $l > d$. The probability $p(i)$ of exactly i intersections is given in Theorem 1. This differs only notationally from results in Kendall and Moran ([3], pp. 73–74). Throughout this paper let $a = l/d$, let $[x]$ denote the greatest integer less than or equal to x and write $f(x) = O(g(x))$ as $x \to x_0$ if $|f(x)| \leqq K|g(x)|$ for some $k > 0$ in a neighborhood of x_0.

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Theorem 1. The number of intersections can range between 0 and $[a]+1 \overset{d}{=} M$. Let the angles θ_i $(0 \leqq \theta_i \leqq \frac{1}{2}\pi)$ be determined by $\cos\theta_i = i/a$. Let $\delta_i \overset{d}{=} (2a \sin\theta_i/\pi) - (2i\theta_i/\pi)$. Then for $[a] \geqq 2$:

$$p(0) = \delta_1 + 1 - (2a/\pi),$$

$$p(i) = \delta_{i-1} + \delta_{i+1} - 2\delta_i \quad \text{for} \quad 1 \leqq i \leqq M-2,$$

$$p(M-1) = \delta_{M-2} - 2\delta_{M-1}; \quad p(M) = \delta_{M-1}.$$

For $[a]=1$, the results for $p(0)$ and $p(M)$ above hold, and $p(1) = (4\theta_1/\pi) + (2a/\pi) - (4a\sin\theta_1/\pi)$.

Let $\{b_i\}_{i=0}^{M}$ be any real numbers. Summation by parts shows:

$$\sum_{i=0}^{M} b_i p_i = b_0(1-(2a/\pi)) + b_1\delta_0 + \sum_{i=1}^{M-1} \Delta^2(b_{i-1})\delta_i. \tag{2.1}$$

Here $\Delta(b_i) \overset{d}{=} b_{i+1} - b_i$ is the differencing operator. (2.1) easily yields $\Sigma_{i=0}^{M} p_i = 1$ as well as $\mu_1 \overset{d}{=} \Sigma_{i=0}^{M} ip_i = (2a/\pi)$. The expression for the mean is frequently derived using an approximation argument ([5], p. 253). Another easy consequence of (2.1) is the following simple form for the distribution function:

$$p(\text{number of crossings} \leqq i) \overset{d}{=} F(i) = 1 - (\delta_i - \delta_{i+1}) \tag{2.2}$$

for $i = 0, 1, \cdots, M-1$.

For higher moments, some approximation is needed.

Theorem 2.

$$\mu_k \overset{d}{=} \sum_{i=0}^{M} i^k p_i = c_k a^k + O(a^{k-3/2})$$

as $a \to \infty$ for $k \geqq 1$ where

$$a = \frac{l}{d}$$

and

$$c_k = \{\Gamma(\tfrac{1}{2}(k+1))\}/\Gamma(\tfrac{1}{2}(k+2))\sqrt{\pi}.$$

Proof. The result is true with no error for $k=1$. The proof for $k=2,3$ requires consideration of special cases. The modifications of the approach used below are straightforward and omitted. Thus assume $k>3$. Using (2.1):

$$\mu_k = \delta_0 + \sum_{i=1}^{M-1} \Delta^2[(i-1)^k]\delta_i.$$

Since $\delta_0 = O(a)$ and it is easy to see

$$\begin{aligned} &\Delta^2[(i-1)^k] = k(k-1)i^{k-2} + O(i^{k-4}) \quad \text{as} \quad i \to \infty, \\ &\mu_k = k(k-1)\sum_{i=1}^{M} i^{k-2}\delta_i + O(a) + O\left(\sum_{i=1}^{M-1} i^{k-4}\delta_i\right) \quad \text{as} \quad i \to \infty,\ a \to \infty. \end{aligned} \tag{2.3}$$

Consider the first sum in (2.3).

$$\sum_{i=1}^{M-1} i^{k-2}\delta_i = \frac{2}{\pi}\sum_{i=1}^{M-1} i^{k-2}\left\{a\left[1-\left(\frac{i}{a}\right)^2\right]^{1/2} - i\cos^{-1}\left(\frac{i}{a}\right)\right\} = \frac{2}{\pi}a^{k-1}\sum_{i=1}^{M-1} f\left(\frac{i}{a}\right)$$

where $f(x) = x^{k-2}(1-x^2)^{1/2} - x^{k-1}\cos^{-1}x$. A slight rewriting of the Euler Maclaurin formula ([4], p. 542) yields, for any twice differentiable g,

$$\begin{aligned} \sum_{i=1}^{M-1} g(i) = \int_0^{M-1} g(t)dt + \frac{1}{2}\Big[g(M-1) + g(0)\Big] + \frac{1}{12}\Big[g'(M-1) - g'(0)\Big] \\ - \int_0^{M-1} p_2(x)g''(x)dx \end{aligned}$$

where $p_2(x)$ is the periodic continuation, with period 1, of the function taking values $x^2/2 - x/2 + 1/12$ on $0 \leqq x \leqq 1$. Taking

$$\begin{aligned} g(x) &= f\left(\frac{x}{a}\right), \\ g'(x) &= \left(\frac{1}{a}\right)\left\{(k-2)\left(\frac{x}{a}\right)^{k-3}\left[1-\left(\frac{x}{a}\right)^2\right]^{1/2} - (k-1)\left(\frac{x}{a}\right)^{k-2}\cos^{-1}\left(\frac{x}{a}\right)\right\}; \\ g''(x) &= \left(\frac{1}{a^2}\right)\left\{(k-2)(k-3)\left(\frac{x}{a}\right)^{k-4}\left[1-\left(\frac{x}{a}\right)^2\right]^{1/2} + \left(\frac{x}{a}\right)^{k-2}\left[1-\left(\frac{x}{a}\right)^2\right]^{-1/2}\right. \\ &\qquad \left. - (k-1)(k-2)\left(\frac{x}{a}\right)^{k-3}\cos^{-1}\left(\frac{x}{a}\right)\right\}. \end{aligned}$$

Making the substitutions leads to:

$$\begin{aligned} \sum_{i=1}^{M-1} f\left(\frac{i}{a}\right) &= \int_0^{M-1} f\left(\frac{t}{a}\right)dt + \frac{1}{2}\left[f\left(\frac{M-1}{a}\right) + f(0)\right] + \frac{1}{12a}\left[f'\left(\frac{M-1}{a}\right) - f'(0)\right] \\ &\quad - \frac{1}{a^2}\int_0^{M-1} p_2(x)f''\left(\frac{x}{a}\right)dx \\ &= a\int_0^1 f(y)dy - a\int_{1-\theta}^1 f(t)dt + O(a^{-1/2}). \end{aligned} \tag{2.4}$$

Here $\theta = O(a^{-1})$ as $a \to \infty$, and we have used the easily verified fact that $f(t) = O\{(1-t)^{1/2}\}$ as $t \to 1$. Using this last bound in the second integral in (2.4) leads to

$$\sum_{i=1}^{M-1} f\left(\frac{i}{a}\right) = a\int_0^1 f(t)dt + O(a^{-1/2}). \tag{2.5}$$

But

$$\int_0^1 f(t)dt = \begin{cases} \frac{1}{2}B\left(\frac{1}{2}(k-1), \frac{3}{2}\right) - 2k^{-1}B\left(\frac{1}{2}(k-1), \frac{1}{2}\right) & \text{for} \quad k \geqq 3 \\ \frac{1}{8}\pi & \text{if} \quad k = 2 \end{cases}$$

where $B(r, s)$ denotes the beta function. Replacing the sum by the integral, using the bound (2.5) for the sum in the error term of (2.3) and simplifying the beta factors leads to the theorem.

Theorem 2 leads to useful numerical approximations to the moments of the distribution of the number of crossings. Table 1 gives a numerical example for the second moment.

TABLE 1

Actual vs. approximate second moment of the number of crossings of a needle of length l for parallel lines distance d apart

l/d	4.5	9.5	14.5	19.5	24.5
Actual	10.30	45.30	105.30	190.30	300.30
Approximate	10.13	45.13	105.13	190.13	300.13

Define the random variable I as the number of crossings of a needle of length l when thrown at random on a plane ruled with parallel lines distance d apart.

Theorem 3. As $a \to \infty$, (I/a) converges in distribution to an arc sine distribution with density

$$f(x) = \begin{cases} \left(\frac{2}{\pi}\right)\frac{1}{(1-x^2)^{1/2}} & \text{for} \quad 0 \leqq x \leqq 1 \\ 0 & \text{elsewhere}. \end{cases}$$

Proof. As $a \to \infty$ the moments of I/a converge to the numbers c_k of Theorem 2. A straightforward computation shows the arc sine distribution has moments c_k. Since all the distributions concerned are constrained to the unit interval, the method of moments is in force and yields the desired result.

Remark. It is also possible to give a geometric proof of Theorem 3. This has the advantages of showing why the arc sine distribution appears as well as yielding a rate of convergence. Once it has fallen, translations of the needle, which preserve its angle to the grid, can only change the number of intersections by one. The number of intersections for a fixed angle θ is $[a\,|\sin\theta|]$ where $[\cdot]$ denotes greatest integer. Thus, letting I be the number of intersections,

$$\left(\frac{I}{a}\right)-\left(\frac{2}{a}\right)\leqq|\sin\theta|\leqq\left(\frac{I}{a}\right)+\left(\frac{2}{a}\right),$$

so, for $0<t<1$,

$$p(0\leqq\sin\theta\leqq t-(2/a))\leqq p((I/a)\leqq t)\leqq p(0\leqq\sin\theta\leqq t+(2/a)).$$

Now $p(0\leqq\sin\theta\leqq x)=4\sin^{-1}x/2\pi$ by symmetry. Finally, for a so large that $0<x-(2/a)<x+(2/a)<1$, $p((I/a)\leqq x)=2\sin^{-1}x/\pi+O(1/a)$ as $a\to\infty$. The constant implicit in the error term may be chosen independent of x for x bounded away from 1.

A reference to the arc sine density appearing in Theorem 3 may be found in [1], p. 527.

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