APPLICATIONS OF MURPHY'S ELEMENTS

BY

PERSI DIACONIS AND CURTIS GREENE

TECHNICAL REPORT NO. 335
SEPTEMBER 1989

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT DMS89-05874

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

APPLICATIONS OF MURPHY'S ELEMENTS

BY

PERSI DIACONIS AND CURTIS GREENE

TECHNICAL REPORT NO. 335

SEPTEMBER 1989

PREPARED UNDER THE AUSPICES

OF

NATIONAL SCIENCE FOUNDATION GRANT DMS89-05874

DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

Applications of Murphy's elements

Persi Diaconis *
Department of Math
Harvard University
Cambridge, MA 02138

Curtis Greene †
Department of Math
Haverford College
Haverford, PA 19041

September 17, 1989

Abstract

In the symmetric group S_n consider the elements

$$R_i = (1,i) + (2,i) + \cdots + (i-1,i), \quad 2 \leq i \leq n$$

Murphy [17] showed that these R_i are commuting elements of the group algebra, and explicitly computed their eigenvalues at irreducible representations. We use Murphy's results to derive character formulae and to derive closed form expressions for random walk problems. We show that the R_i generate a maximal commutative subalgebra of the group algebra containing the center. Finally, we suggest an underlying reason for Murphy's results which in turn suggests generalizations.

1 Introduction

Alfred Young derived three distinct forms for the irreducible representations of the symmetric group S_n. Young's natural form has integer entries and a combinatorial description which can be neatly presented using Specht modules, as in James [12]. Young's seminormal form has rational entries but

*Research supported in part by N.S.F. Grant DMS 89-05874
†Research supported in part by N.S.F. Grant DMS 87-06093

restricts from S_n to S_{n-1} in block diagonal form without change of basis. A modern construction of Young's seminormal form appears in James-Kerber ([13], Section 3.3). Young's orthogonal form gives orthogonal matrices and block diagonal restriction. It is obtained by an explicit change of basis from the seminormal form ([13], Section 3.4).

Murphy [17] gives a new derivation of the seminormal form from the natural form. A central part of his construction is based on the elements

$$R_i = \sum_{1 \leq j < i} (j, i) \qquad 2 \leq i \leq n \tag{1}$$

Murphy showed the R_i commute and obtained explicit diagonal forms for these elements as linear operators on the group algebra.

To state Murphy's results in a form useful to present applications, order the standard Young tableaux (SYT) by their last letter order. Thus if $\lambda = \{3, 2\}$, the five standard Young tableaux are

$$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array} < \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \end{array} < \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \end{array} < \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array} < \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array}$$

In general, if n is in a lower row, a tableau is further down in the order. If n is in the same row, delete and compare $n - 1$.

The number of standard tableaux of shape λ will be denoted by $f(\lambda)$. Let $T_1 < T_2 < \cdots T_{f(\lambda)}$ be the standard tableaux of shape λ. Young's seminormal form $\rho_\lambda(\pi)$ is an $f(\lambda) \times f(\lambda)$ matrix indexed by these tableaux for each $\pi \in S_n$. Let the transform of R_i at ρ_λ be defined by

$$\hat{R}_i = \sum_{1 \leq j < i} \rho_\lambda((j, i)) \qquad 2 \leq i \leq n$$

Theorem 1.1 (Murphy [17]) *In the symmetric group S_n the R_i defined by (1) are commutative elements of the group algebra with*

$$\hat{R}_i = \mathrm{Diag}(c_1(i), c_2(i), \ldots, c_{f(\lambda)}(i)) \tag{2}$$

in Young's seminormal form. Here $c_j(i) = \mathrm{col}(i) - \mathrm{row}(i)$ in the jth SYT.

Remarks. $c_j(i)$ will be called the *content* of i in the jth standard tableau. As an example, when $n = 5$ and $\lambda = \{3, 2\}$

$$\hat{R}_4 = \text{Diag}(0, 0, 2, 2, -1)$$

The $c_j(i)$ are eigenvalues of the transform of R_i in any basis. It is straightforward to argue that R_i commute, by a combinatorial argument. This fact also follows from the diagonal form of their transforms.

The present paper applies Theorem 1.1 and attempts to give some motivation. In Section 2 we show how it yields closed form expressions for the characters of ρ_λ at simple conjugacy classes. In section 3 we show how it allows explicit Fourier analysis of random walk problems. Section 4 presents a view of Murphy's result which makes it look natural (to us) and suggests appropriate variations for other groups. Section 5 studies the algebra generated by the R_i. The main result here shows that this algebra is a maximal commutative subalgebra of the group algebra $\mathbf{Q}S_n$, of dimension $\sum f(\lambda)$.

Acknowledgment We thank Mehrdad Shahshahani, Richard Stong, and Hansmartin Zeuner for discussions as this work progressed.

2 Formulas for characters

This section derives closed form expressions for characters $\chi_\lambda(\pi)$, when π is a transposition, 3-cycle, product of two 2-cycles, n-cycle, and $(n-1)$-cycle. It also gives a proof of an identity for Schur functions due to Frobenius. Murphy's elements enter through the following result.

Proposition 2.1 *Let R_i be defined by (1). The kth elementary symmetric function in $R_2, R_3, \ldots, R_n$ is the sum over all conjugacy classes with $n-k$ cycles. Thus for example*

$$\begin{aligned}
\textstyle\sum_{i=2}^n R_i &= \textstyle\sum\{\text{transpositions}\} \\
\textstyle\sum_{2\le i<j\le n} R_i R_j &= \textstyle\sum\{\text{3-cycles}\} + \sum\{\text{products of 2 disjoint 2-cycles}\} \\
\textstyle\prod_{i=2}^n R_i &= \textstyle\sum\{n\text{-cycles}\}
\end{aligned}$$

Proof: Let q be an indeterminate. Then

$$\prod_{i=2}^{n} (I + qR_i) = I + q\sum_i R_i + q^2 \sum_{i<j} R_i R_j + \cdots + q^{n-1} \prod_{i=2}^{n} R_i \qquad (3)$$

We need to show that

$$\prod_{i=2}^{n} (I + qR_i) = \sum_{\pi \in S_n} \pi q^{n-\gamma(\pi)} \tag{4}$$

where $\gamma(\pi)$ denotes the number of cycles in π. Assume this fact is true in S_{n-1}, which we identify with the set of all permutations in S_n fixing n. Coset representatives for S_{n-1} in S_n are $(1,n),(2,n),\ldots(n-1,n)$. Furthermore, multiplying any permutation in S_{n-1} by (i,n) decreases the number of cycles by 1. It follows that

$$\sum_{\pi \in S_n} \pi q^{n-\gamma(\pi)} \quad = \quad qR_n \sum_{\pi \in S_{n-1}} \pi q^{n-1-\gamma(\pi)}$$

Now the result follows by induction.

Corollary 2.2 *Let $\chi_\lambda(2)$ be the character of the irreducible representation determined by λ at the transposition τ. Then*

$$\chi_\lambda(2) \;=\; \frac{f(\lambda)}{\binom{n}{2}}\left[\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2}\right] \;=\; \frac{f(\lambda)}{n(n-1)}\sum \left[\lambda_i^2 - (2i-1)\lambda_i\right]$$

Proof: Let ρ_λ be the corresponding irreducible representation. From Schur's Lemma and Proposition 2.1

$$\sum_{i=2}^{n} \hat{R}_i \;=\; \sum_{\text{2-cycles } \tau} \rho_\lambda(\tau) \;=\; cI, \quad \text{with} \quad c = \frac{\binom{n}{2}\chi_\lambda(\tau)}{f(\lambda)}$$

Indeed, the matrix $M = \sum \rho_\lambda(\tau)$ is invariant under the action of S_n, that is, $\rho_\lambda(\sigma^{-1})M\rho_\lambda(\sigma) = M$ for all $\sigma \in S_n$. It follows that $M = cI$, and taking traces yields the value of c.

On the other hand, c can be computed by considering the $(1,1)$ element in $\sum \hat{R}_i$. This is equal to $\sum \mathrm{col}(i) - \mathrm{row}(i)$, as i runs from 2 through n in the first SYT:

$$T_1 \;=\; \begin{matrix} 1 & \lambda'_1+1 & \lambda'_1+\lambda'_2+1 & \cdots \\ 2 & \lambda'_1+2 & \cdots & \cdots \\ 3 & \vdots & & \\ \vdots & \lambda'_2 & & \\ \lambda'_1 & & & \end{matrix}$$

Thus the first tableau T_1 has entries $1, 2, 3, \ldots$ down the columns in order, and λ' denotes conjugation. For this (or any other tableaux), it is clear that

$$c = \sum_i c(i) = \sum_i \binom{\lambda_i}{2} - (i-1)\lambda_i = \frac{1}{2}\sum_i \left[\lambda_i^2 - (2i-1)\lambda_i\right] \qquad (5)$$

Remarks: This formula for $\chi_\lambda(2)$ has many applications. It was used to show the non-existence of perfect codes in the symmetric group by Rothaus and Thompson [18]. Diaconis and Shahshahani [5] use it to study random walks based on randomly transposing pair of cards. Diaconis and Graham [4] use it to determine when Radon transforms on S_n are invertible. These applications and others are reviewed in [2], Section 3-D.

Corollary 2.3 *Let $\chi_\lambda(3)$ be the character of the irreducible representation determined by λ at a 3-cycle. Then*

$$\chi_\lambda(3) = \frac{3f(\lambda)}{n(n-1)(n-2)} \left[\sum_j \left(\lambda_j(j^2 - j + 1) + 3\binom{\lambda_j}{2} + 2\binom{\lambda_j}{3} - j\lambda_j^2\right) - \binom{n}{2}\right]$$

Proof: It is easy to see that

$$\sum_2^n R_i^2 = \binom{n}{2} I + \sum\{\text{3-cycles}\} \qquad (6)$$

since every 3-cycle can be uniquely written as $(i,k)(j,k)$ with $k > i, j$. Further, $\sum_2^n R_i^2$ is in the center of the group algebra, so Schur's Lemma gives

$$\sum_2^n \hat{R}_i^2 = c\,I$$

with

$$c\,f(\lambda) = \binom{n}{2} f(\lambda) + 2\binom{n}{3}\chi_\lambda(3)$$

As before, c can be computed by looking at the (1,1) entry. Writing $c(i) = \text{col}(i) - \text{row}(i)$ in the first tableau, we get $c = \sum_i c(i)^2$. Finally

$$\chi_\lambda(\text{3-cycle}) = \frac{3f(\lambda)}{n(n-1)(n-2)} \left(\sum_i c(i)^2 - \binom{n}{2}\right)$$

Summing $c(i)^2$ over T_1 as in Corollary 2.2 gives

$$\sum c(i)^2 = \sum_j \left(\lambda_j(j^2 - j + 1) + 3\binom{\lambda_j}{2} + 2\binom{\lambda_j}{3} - j\lambda_j^2\right) \quad (7)$$

Corollary 2.4 *Let $\chi_\lambda(2,2)$ be the character of the irreducible representation determined by λ at the product of two disjoint 2-cycles. Then*

$$\chi_\lambda(2,2) = \frac{f(\lambda)}{6\binom{n}{4}}((\sum c(i))^2 - 3\sum c(i)^2 + 2\binom{n}{2})$$

with $\sum c(i)$ given as in (5) and $\sum c(i)^2$ given as in (7).

Proof: Arguing as in Corollary 2.3 it is easy to show that

$$(\sum_2^n R_i)^2 = \binom{n}{2} I + 3\sum\{\text{3-cycles}\} + 2\sum\{\text{two 2-cycles}\} \quad (8)$$

Solving (6) and (8) gives

$$\sum \text{two 2-cycles} = \frac{1}{2}(\sum R_i)^2 - \frac{3}{2}(\sum R_i^2) + \binom{n}{2} I$$

Computing characters as above yields

$$\chi_\lambda(2,2) = \frac{f(\lambda)}{6\binom{n}{4}}((\sum c(i))^2 - 3\sum c(i)^2 + 2\binom{n}{2})$$

Corollary 2.5 *Let $\chi_\lambda(n)$ be the character of the irreducible representation determined by λ at an n-cycle. Then*

$$\chi_\lambda(n) = \begin{cases} (-1)^{\lambda'_1 - 1} & \text{if } \lambda \text{ is a hook} \\ 0 & \text{else} \end{cases}$$

with λ' the partition conjugate to λ.

Proof: From Proposition 2.1 we have

$$\prod_{i=2}^n \hat{R}_i = \sum_{n\text{-cycles } \gamma} \rho_\lambda(\gamma) = (n-1)!\,\frac{\chi_\lambda(n)}{f(\lambda)}$$

Consider the $(1,1)$ term of the product. Clearly this is zero unless λ is a hook — indeed some element i in $2,3,\ldots,n$ occupies the 2-2 position, and so for this element $\mathrm{col}(i) - \mathrm{row}(i) = 0$. For a hook shape the first-ordered tableau appears as

$$T_1 = \begin{array}{llll} 1 & \lambda_1'+1 & \cdots & n \\ 2 \\ 3 \\ \vdots \\ \lambda_1' \end{array}$$

This shows the product has $(1,1)$ entry

$$(1-2)(1-3)\cdots(1-\lambda_1')(2-1)(3-1)\cdots(\lambda_1-1) = (-1)^{\lambda_1'-1}(\lambda_1'-1)!(\lambda_1-1)!$$

For a hook shape λ,

$$f(\lambda) = \frac{(n-1)!}{(\lambda_1'-1)!(\lambda_1-1)!}.$$

and the proof is complete.

Remarks.

1. The character formulas in this section can be obtained by a variety of other methods. For example, Ingram [11] uses the Murnaghan-Nakayama recursion to derive explicit formulas for $\chi_\lambda(2)$, $\chi_\lambda(3)$, $\chi_\lambda(4)$, $\chi_\lambda(5)$, $\chi_\lambda(2,2)$, $\chi_\lambda(3,2)$, $\chi_\lambda(4,2)$, $\chi_\lambda(3,3)$, and $\chi_\lambda(2,2,2)$. Our derivations are simpler, and perhaps more "natural", but depend on knowing the eigenvalues of the R_i's explicitly.

2. From Corollary 2.5 it is easy to derive the character at an $(n-1)$-cycle π — say fixing n. Consider $S_{n-1} \subseteq S_n$ as all permutations fixing n. In Young's seminormal form,

$$\rho_\lambda(\pi) = \begin{pmatrix} \rho_{\lambda^1}(\pi) & & \\ & \ddots & \\ & & \rho_{\lambda^d}(\pi) \end{pmatrix}$$

where $\lambda^1, \lambda^2, \ldots$ are the partitions of $n-1$ derived from the branching rule, i.e. by deleting all possible corner cells (see [13] p. 118). The previous formula can be applied to each component matrix. The only nonzero cases have λ a hook, or $\lambda = \{a, 2, 1, 1, 1, \ldots\}$. The first case gives zero, the second gives $(-1)^{\#1\text{'s}+1}$. Thus

$$\chi_\lambda(n-1) = \begin{cases} 0 & \text{unless } \lambda = \{a,2,1,1,1,\ldots\} \\ (-1)^{\#\,1\text{'s}+1} & \text{if } \lambda = \{a,2,1,1,1,\ldots\} \end{cases}$$

Using this with Corollary 2.5 allows us to demonstrate an observation of Serre [19]: *the product* $\chi_\lambda(n)\chi_\lambda(n-1)$ *equals* 0 *unless* λ *is the trivial representation or the alternating representation.*

The identities of this section are developed further in Section 5 where it is shown that any conjugacy class can be written as a symmetric polynomial in the R_i. To conclude this section we use Murphy's elements to prove an identity of Frobenius relating to Schur functions.

The Schur functions are a well known basis for the homogeneous symmetric functions. Their properties are described and developed in Stanley [20] and Macdonald [14]. One convenient definition:

$$s_\lambda(x_1, x_2, \ldots, x_m) = \sum_{\rho \vdash n} \frac{\chi_\lambda(\rho)}{z_\rho} p_\rho(x_1, x_2, \ldots, x_m) \tag{9}$$

where p_ρ is the power sum symmetric function $p_\rho = p_{\rho_1} p_{\rho_2} \cdots p_{\rho_n}$ ($p_j = \sum_i x_i^j$), $\chi_\lambda(\rho)$ is the character of the λth irreducible representation at the ρth conjugacy class, and $z_\rho = \prod i^{a_i} a_i!$ where ρ has a_i parts equal to i. Finally, λ is any partition of $n = \lambda_1 + \lambda_2 + \cdots$ into m or fewer parts.

Corollary 2.6 (Frobenius [8])

$$s_\lambda(1, 1, \ldots, 1) \;=\; \prod_{x \in \lambda} \frac{m + c(x)}{h(x)} \tag{10}$$

where the product is over all cells x *in the Ferrers diagram of* λ, $h(x)$ *is the hook length associated with cell* x, *and* $c(x)$ *is the content of* x, $c(x) = \mathrm{col}(x) - \mathrm{row}(x)$.

Proof: It follows from (9) that

$$\begin{aligned} s_\lambda(1,1,\ldots,1) &= \sum_{\rho \vdash n} \frac{\chi_\lambda(\rho)}{z_\rho} m^{\gamma(\rho)} \\ &= \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi) m^{\gamma(\pi)} \end{aligned}$$

where $\gamma(\rho)$ denotes the number of parts of ρ and $\gamma(\pi)$ denotes the number of cycles of π. We will show that this last expression equals the right hand side of (10).

Substituting $q = 1/n$ in identity (4) of Proposition 2.1 gives

$$\sum_{\pi\in S_n} \pi m^{\gamma(\pi)} \quad = \quad m\prod_{2}^{n}(mI + R_i) \tag{11}$$

Taking traces in the λth representation, we obtain on the left

$$\sum_{\pi\in S_n} \chi_\lambda(\pi) m^{\gamma(\pi)} \quad = \quad n! s_\lambda(1,1,\ldots,1) \tag{12}$$

On the right hand side each term $(mI + R_i)$ transforms to a diagonal matrix, and the product equals a constant times the identity. Hence the trace equals $f(\lambda)$ times the $(1,1)$ entry, which equals

$$f(\lambda)\, m \prod_{i=2}^{n}(m + c(i)) \quad = \quad \frac{n!}{\prod_{x\in\lambda} h(x)} \prod_{x\in\lambda}(m + c(x)) \tag{13}$$

Here $c(i)$ denotes the content of the cell containing i in the first (or any other) standard tableau. Equating (12) and (13) completes the proof.
Remarks. One can view (4) as a "lifting" of identity (10) to the group algebra. According to Stanley ([20], Part 2, p. 263), Corollary 2.6 was first proved by Frobenius ([8], section 3). MacMahon [15] gave a combinatorial proof: the left hand side counts the number of column strict plane partitions of shape λ with largest entry at most n. Macdonald ([14], p. 27, 28) gives a direct proof using the determinant definition of Schur functions.

3 An application to card shuffling

Consider n cards face down in a row on a table. Suppose the cards are numbered $1, 2, \ldots, n$, from left to right. The cards are repeatedly mixed by randomly transposing pairs according to the following scheme: the right hand picks a card with chance $\omega(i)$ of the ith position. Here $\omega(i) \geq 0, \sum\omega(i) = 1$ is a fixed probability on $\{1, 2, \ldots, n\}$. Then the left hand picks a card at random (uniformly) between position 1 and the right hand card. These cards are then transposed. This can be described by the following probability distribution

$$
\begin{aligned}
p(\mathrm{id}) &= \omega(1) \\
p((i,j)) &= \tfrac{\omega(j)}{j-1} \quad 1 \le i < j \\
p(\pi) &= 0 \quad \text{otherwise}
\end{aligned}
$$

Two special cases have been carefully considered previously. Diaconis and Shahshahani [5] worked with random transpositions. Flatto, Odlyzko, and Wales [7] worked with randomly transposing a card with card 1. In both cases, the diagonal nature of the Fourier transform permits complete answers to the standard questions. For example, it takes $(1/2)n\log n + cn$ random transpositions to mix things up ([5]). In takes $n \log n + cn$ transpositions with 1 to mix things up ([3]). It takes order $n!$ steps to return to the starting arrangement ([7]). It takes order $n! \log n!$ steps to hit every arrangement ([16]).

The formulas above allow these questions to be studied for any system of weights $\omega(i)$. Thus if ρ_λ is an irreducible representation in Young's seminormal form then

$$\hat{p}(\rho_\lambda) \stackrel{\text{def}}{=} \sum p(\pi)\rho_\lambda(\pi)$$

is diagonal, with ith element

$$\omega(1) + \sum_{j=2}^{n} \frac{\omega(j)}{j-1} c_i(j)$$

where $c_i(j)$ is as defined in (2). From here it is possible to carry out the computations more or less as described by the authors cited above.

As discussed by Diaconis ([2] Chapter 3-B) the availability of a closed form for the Fourier transform has application to the existence of perfect codes ([1], [18]), to the inversion of Radon transforms ([4]), and to graph labeling problems. Each of these applications can be extended to varying weights.

4 A general remark

The properties of Murphy's elements seem mysterious when first encountered. In this section we present several results which make the R_i's seem like a fairly natural construction, and suggest extensions to other groups.

Definition 4.1 *Let $G \supseteq H \supseteq K$ be a tower of subgroups, and let h be an element of $H - K$. Let*

$$R_h = \sum_{k \in K}^{*} k^{-1}hk$$

*where the * denotes that only distinct terms in the sum are taken.*

Definition 4.2 *Let $G \supseteq H \supseteq K$, and let ρ be a representation of G on a vector space V. A basis B of V is said to* block-reduce *ρ with respect to H and K if $\rho|H$ is represented by a matrix in which the irreducible constituents of $\rho|H$ appear as diagonal blocks, and the same is true of $\rho|K$.*

Proposition 4.3 *Let $G \supseteq H \supseteq K$ as above, with $h \in H - K$. Suppose that the action of K on H by conjugation splits the H-orbit of h into two K-orbits, one of which lies in K. Let ρ be a representation of G on V, and let B be a basis which block-reduces ρ with respect to H and K. Then for R_h as defined above, $\hat{R}_h = \rho(R_h)$ is diagonal.*

Proof: Let h and k be representatives of the two K-orbits of h. Then

$$R_h = C_h^{(H)} - C_k^{(K)}$$

where $C_h^{(H)}$ and $C_k^{(K)}$ denote the class sums of h and k in H and K respectively. It follows from Schur's lemma and the block diagonal nature of ρ that both $\rho(C_h^{(H)})$ and $\rho(C_k^{(K)})$ are diagonal matrices, and the conclusion follows immediately.

Example 1. Consider $S_n \supset S_j \supset S_{j-1}$, and take $h = (1, j)$. Then

$$R_h = R_j = (1,j) + (2,j) + \cdots + (j-1,j)$$

and Young's seminormal form as constructed restricts in the required block diagonal fashion. Thus $\hat{R}_j(\rho)$ is a diagonal matrix.

The next proposition shows that the conditions on h in Proposition 4.3 may be weakened, if we assume a somewhat stronger condition on ρ.

Proposition 4.4 *Let $G \supseteq H \supseteq K$, and let ρ be a representation of G on V such that for every irreducible η occurring in $\rho|H$, the restriction $\eta|K$ is multiplicity free. Let B be a basis of V which block-reduces ρ with respect to H and K. Let R be an element of the group algebra whose support lies in H and is K-conjugacy invariant. Then $\hat{R} = \rho(R)$ is a diagonal matrix.*

Proof: Clearly $\hat{R}$ is block diagonal with respect to the irreducible constituents of $\rho|H$, since the coefficients of R vanish off H. Further, Schur's Lemma yields that the H blocks of $\hat{R}$ must themselves be block diagonal with respect to their K-constituents. (Here we need the fact that within each H-block the K-constituents are distinct.) Finally, each K-block is a constant times the identity, since R is K-invariant. Hence $\hat{R}$ is diagonal, and the proof is complete.

Example 2. Take, as above, $S_n \supset S_j \supset S_{j-1}$. Let $\pi \in S_j - S_{j-1}$. Let

$$R_\pi = \sum_{\sigma \in S_{j-1}}^{*} \sigma^{-1}\pi\sigma$$

as in Definition 4.1. By the branching theorem, any irreducible representation of S_j restricts in a multiplicity free way to S_{j-1}. Thus the hypotheses of Proposition 4.4 are satisfied, and it follows that all $\hat{R}_\pi$ are diagonal in Young's seminormal form. We record this formally:

Proposition 4.5 *Let $S_n \supset S_{n-1} \supset S_{n-2} \supset \cdots \supset \{\text{id}\}$ with S_k fixing $k+1, \ldots, n$. For each $\pi \in S_n$ define*

$$R_\pi = \sum_{\sigma \in S_{L-1}}^{*} \sigma^{-1}\pi\sigma$$

where L is the largest index such that $\pi \in S_L - S_{L-1}$ and the sum is over distinct terms in the S_{L-1} orbit of π. Then all such R_π commute and have diagonal transforms at all irreducible representations given in Young's seminormal form.

Remarks.

1. In Section 5 we show that the R_π of Proposition 4.5 are in the algebra generated by the R_i. Of course Murphy's elements are the simplest R_π in a sense, being $R_{(1,j)} = R_j$.

2. The orbits of S_{n-1} on S_n partition the S_n conjugacy classes, and are easy to describe. If $\pi \in S_n$ has cycle length indicator $(a_1(\pi), a_2(\pi), \ldots, a_n(\pi))$ with $a_i(\pi)$ the number of i-cycles in π, its S_{n-1} orbit consists of all permutations of cycle type $(a_1, a_2, \ldots, a_n)$ with n in the same size cycle as it is in π and $1, 2, \ldots, n-1$ distributed arbitrarily. Thus the class sum with indicator $(a_1, a_2, \ldots, a_n)$ in S_n splits into k orbits under S_{n-1}, where k is the number of non-zero a_i.

3. The hyperoctahedral group B_n also has the property that any irreducible representation of B_n restricted to B_{n-1} splits in a multiplicity free way. Thus a similar theory can be developed for B_n. Indeed the elements

$$\begin{aligned} P_k &= \bar{k} \\ Q_k &= \sum_{i<k} (i, k) + (\bar{i}, \bar{k}) \end{aligned}$$

can be viewed as "Murphy's elements" for B_n, in the sense that they commute in $\mathbf{Q}B_n$ and satisfy the group algebra identity

$$B_n = \prod_{k=1}^{n} (I + P_k + Q_k)$$

We plan to develop these ideas further in another work.

4. Following work of Hirschman [10] and Dunkl [6], Greenhalgh [9] studied commutative algebras of functions on a finite group G which were conjugacy invariant under a subgroup H and bi-invariant under a subgroup K with H normalizing K. An example is $S_n \supset S_k \times S_{n-k} \supset S_k$. The techniques involved are similar to the considerations of this section but the exact connection is unclear: we do not have a symmetry characterization of the algebra generated by Murphy's elements, and R_k is conjugacy invariant under $S_{k-1} \times S_{n-k}$ and bi-invariant under S_{n-k}.

5. In previous work, Flatto, Odlyzko and Wales [7] derived the diagonal form of $\hat{R}_n$ directly. An idea suggested by Hansmartin Zeuner can be used to derive the diagonal values of $\hat{R}(\pi)$ from known values of $\chi_\lambda(2)$. For example, consider

$$\sum_{1 \le i < j \le n} (i, j) = \sum_{i<n} (i, n) + \sum_{1 \le i < j \le n-1} (i, j)$$

The transform of the left hand side is

$$\frac{\binom{n}{2}\chi_\lambda(\tau)}{f(\lambda)} I = \sum_{x \in \lambda} c(x)$$

in any basis. The first sum is what we want. The second sum is over S_{n-1}. Choose a basis so this is block diagonal restricted to S_{n-1} (for example, as in Young's seminormal form). We know that the blocks are constants times I, the constant being

$$\frac{\binom{n-1}{2}\chi_{\lambda^i}(\tau)}{f(\lambda^i)} = \sum_{x \in \lambda^i} c(x)$$

where λ^i denotes the ith piece under branching. It follows that $R_n = \sum_{i<n}(i,n)$ transforms to a diagonal matrix with eigenvalues of the form

$$\sum_{x \in \lambda} c(x) - \sum_{x \in \lambda^i} c(x) = c(m)$$

Thus $\hat{R}_n(\rho)$ is as described in Theorem 1.1. Using induction gives the values of the other $\hat{R}_i(\rho)$ as known.

It seems natural to try to derive other diagonal forms similarly. Thus

$$\sum(i,j,k) = \sum_{i,j<n}(i,j,n) + \sum_{i,j,k<n}(i,j,k)$$

gives the diagonal values for $\sum(i,j,n) = R_{(1,2,n)}$.

This approach works nicely for a single j-cycle. For more complex classes, say the product of 2-cycle and a 3-cycle, the right hand side doesn't split in such a simple way.

5 The Murphy Algebra

In this section we will study the subalgebra $\mathcal{M} \subseteq \mathbf{Q}S_n$ generated by the R_i's. It is well known that the group algebra $\mathbf{Q}S_n$ is isomorphic to a direct sum of matrix algebras, that is

$$\mathbf{Q}S_n \approx \bigoplus_\lambda \mathcal{A}_\lambda$$

Young's seminormal form gives one such isomorphism. Let

$$\mathcal{D} = \bigoplus_{\lambda} \mathcal{D}_\lambda$$

be the subalgebra of $\mathbf{Q}S_n$ generated by elements corresponding to matrices which are diagonal in Young's seminormal form. Clearly $\mathcal{D}$ is semisimple and

$$\dim(\mathcal{D}) = \sum_{\lambda} f(\lambda)$$

Further, let $\mathcal{Z}$ denote the center of $\mathbf{Q}S_n$. It is elementary and well known that $\mathcal{Z}$ is spanned by the class functions in S_n. Another basis for $\mathcal{Z}$ consists of the characters $\sum_\pi \chi_\lambda(\pi)\pi$, which, under the isomorphism defined above correspond to constant matrices in $\mathcal{A}_\lambda$. Clearly $\dim(\mathcal{Z}) = p(n)$, the number of partitions of n.

Theorem 5.1

1. *$\mathcal{M} = \mathcal{D}$, that is, $\mathcal{D}$ is generated by the R_i. In particular, $\mathcal{M}$ is a maximal commutative subalgebra of $\mathbf{Q}S_n$.*
2. *$\mathcal{Z} \subset \mathcal{M}$ consists of those elements which can be expressed as symmetric polynomials in the R_i.*

As a consequence of part (1) we obtain

Corollary 5.2 *The elements R_π defined in Proposition 4.5 are in $\mathcal{M}$.*

In simple cases this is easy to see, for example,

$$R_{(1,2,n)} = R_n^2 - n\,I$$

However we do not know a simple argument which proves this result in general.

Remark. It is straightforward that the primitive idempotents of the algebra $\mathcal{M}$, thought of as functions on the group, are given by the functions

$$\pi \longrightarrow \frac{f(\lambda)}{n!}\rho_\lambda(\pi^{-1})_{ii}$$

Here λ ranges over partitions of n, $1 \le i \le f(\lambda)$, and ρ_λ is a representation in Young's seminormal form. Indeed these are just the image under the Fourier transform of the primitive idempotents in the diagonal matrices. We find it an interesting problem to express these directly in terms of the R_i.

Proof of Theorem 5.1 We know that for each λ the matrices $\hat{R}_i = \rho_\lambda(R_i), i = 2, \ldots, n$ are diagonal. It will suffice to show that these matrices generate the full diagonal subalgebra of $f(\lambda) \times f(\lambda)$ matrices, for each λ. Let T_i and T_j be distinct standard tableaux of shape λ. One can show that there exists an m such that $c_i(m) \neq c_j(m)$, in other words is T_i and T_j are distinguished by their contents (see [13], Section 3.3). Now fix i and for each $j \neq i$ let m_i be such that $c_i(m_i) \neq c_j(m_i)$. Then for each $j \neq i$

$$\rho_\lambda(R_{m_i}) - c_j(m_i)I$$

is a matrix with (i,i) entry nonzero and (j,j) entry zero. Thus

$$\prod_{\substack{1 \le j \le f(\lambda) \\ j \neq i}} \rho_\lambda(R_{m_i}) - c_j(m_i)I$$

is a matrix with (i,i) nonzero and all other entries zero. These matrices span the diagonal subalgebra of A_λ, and since each is clearly in the image of $\mathcal{M}$ the proof of part (1) is complete.

To prove part (2) we will prove a somewhat stronger result. The arguments generalize calculations already carried out for simple cases in Section 2. In order to state the result we need a notation for class sums. If μ is a partition of n, let

$$[\mu] = \sum \{\sigma \mid \text{type } \sigma = \mu\}$$

In this notation we also suppress 1's (which correspond to fixed points) in all cases except the identity class. Thus

$$\begin{aligned} [3] &= \textstyle\sum\{\text{3-cycles}\} \\ [2,2] &= \textstyle\sum\{\text{2 disjoint 2-cycles}\} \end{aligned}$$

and so forth. We also need the following definitions: if $\mu = \{\mu_1 \ge \mu_2 \ge \cdots \ge \mu_k\}$, let

$$\begin{aligned} \mu^+ &= \{\mu_1 + 1 \ge \mu_2 + 1 \ge \cdots \ge \mu_k + 1\} \\ \mu^- &= \{\mu_1 - 1 \ge \mu_2 - 1 \ge \cdots \ge \mu_k - 1\} \end{aligned}$$

and let

$$\operatorname{rank}(\mu) = \sum_i \mu_i - 1 = |\mu^-|$$

If σ is a permutation of cycle type λ, define $\operatorname{rank}(\sigma) = \operatorname{rank}(\lambda)$. The main result, which implies part (2) of Theorem 5.1 is the following.

Theorem 5.3 *For every partition λ with $|\lambda| = n_0 \leq n$, there exists a symmetric polynomial ϕ of degree at most* $\operatorname{rank}(\lambda)$ *such that*

$$[\lambda] = \phi(R_2, R_3, \ldots, R_n)$$

Examples. The following formulas were obtained in deriving the results of Section 2:

$$\begin{aligned}
[2] &= e_1(R_2, R_3, \ldots, R_n) = p_1(R_2, R_3, \ldots, R_n) \\
[3] &= p_2(R_2, R_3, \ldots, R_n) - \binom{n}{2} I \\
[2,2] &= \tfrac{1}{2} P_1(R_2, R_3, \ldots, R_n)^2 - \tfrac{3}{2} P_2(R_2, R_3, \ldots, R_n) + \binom{n}{2} I \\
[n] &= e_{n-1}(R_2, R_3, \ldots, R_n)
\end{aligned}$$

Here e_k denotes as usual the kth elementary symmetric polynomial, and p_k denotes the kth power sum symmetric polynomial, $p_k(x_1, \ldots, x_n) = \sum_1^n x^k$. Our goal will be to extend these results.

Lemma 5.4

1. *If $\sigma \in S_n$ has cycle type μ^+ (here we assume that the 1's are suppressed), then*

$$\begin{aligned}
|\mu^+| &= \#\text{ elements moved by } \sigma \\
|\mu| &= \operatorname{rank}(\sigma) = n - \#\text{ cycles of } \sigma
\end{aligned}$$

2. *If α, β are permutations, then* $\operatorname{rank}(\alpha\beta) \leq \operatorname{rank}(\alpha) + \operatorname{rank}(\beta)$.

Lemma 5.5 *Let $P_k = p_k(R_2, R_3, \ldots, R_n) = \sum_2^n R_i^k$. Then*

$$P_k \;=\; [k+1] + \sum_\theta u_\theta \,[\theta] \tag{14}$$

where the u_θ's are constants, and the sum is over classes $[\theta]$ such that $|\theta| < k+1$ (less than $k+1$ elements are moved) and $\text{rank}(\theta) \leq k$.

Proof: Every $(k+1)$-cycle has a unique decomposition $(a_1, i)(a_2, i)\cdots(a_k, i)$, with the a_i's distinct and less than i. This proves that the summand $[k+1]$ occurs with coeffient 1. Every other term in the expansion of R_i^k has at least one repeated transposition, and hence moves at most k elements. Finally, every term is the product of k transpositions, and hence has rank $\leq k$, by Lemma 5.4.

Lemma 5.6 *Let $\mu = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k\}$, with $|\mu^+| = \sum_i(\mu_i + 1) = n_0 \leq n$. Define $P_\mu = P_{\mu_1} P_{\mu_2} \cdots P_{\mu_k}$. Then*

$$P_\mu \;=\; c\,[\mu^+] \;+\; \sum_\theta v_\theta [\theta] \tag{15}$$

where c is a nonzero constant, and the sum is over classes $[\theta]$ such that $|\theta| < n_0$ and $\text{rank}(\theta) \leq |\mu|$.

Proof: Again, the summand $[\mu^+]$ arises from products in which the terms are disjoint $(\mu_i + 1)$-cycles. In every other term of the expansion, some factor moves fewer than $\mu_i + 1$ elements, and hence the product moves fewer than $|\mu^+| = n_0$ elements. Finally, every term appearing in P_{μ_i} has rank $\leq |\mu_i|$, by Lemma 5.5, and hence and the terms in $P_{\mu_1} P_{\mu_2} \cdots P_{\mu_k}$ have rank $\leq \sum \mu_i$ by Lemma 5.4.

Proof of Theorem 5.3 Linearly order the partitions of all integers i with $0 \leq i \leq n-1$ so that $\lambda < \mu$ if $|\lambda| < |\mu|$, or if $|\lambda| = |\mu|$ and $|\lambda^+| < |\mu^+|$. Let $\mathcal{P}$ be the matrix indexed by partitions μ, ν such that $\mathcal{P}_{\mu,\nu}$ is the coefficient of $[\nu^+]$ in the expansion of P_μ. By lemma 5.6, this matrix is triangular, and the diagonal entries $\mathcal{P}_{\mu,\mu}$ are nonzero provided $|\mu^+| \leq n$. By hypothesis, $|\lambda| \leq n$, hence we can solve for $[\lambda]$ and the proof is complete.

Illustration. Here are the expansions of P_μ for $|\mu| \le 3$:

$$\begin{aligned}
P_0 &= I \\
P_1 &= [2] \\
P_2 &= \binom{n}{2} I + [3] \\
P_{1,1} &= \binom{n}{2} I + 3[3] + 2[2,2] \\
P_3 &= (2n-3)[2] + [4] \\
P_{2,1} &= (\tfrac{1}{2}n^2 + \tfrac{3}{2}n - 4)[2] + 4[4] + [3,2] \\
P_{1,1,1} &= (\tfrac{3}{2}n^2 + \tfrac{1}{2}n - 6)[2] + 16[4] + 9[3,2] + 6[2,2,2]
\end{aligned}$$

And here are the corresponding expansions of the class sums:

$$\begin{aligned}
I &= P_0 \\
[2] &= P_1 \\
[3] &= -\binom{n}{2} I + P_2 \\
[2,2] &= \binom{n}{2} P_0 - \tfrac{3}{2} P_2 + \tfrac{1}{2} P_{1,1} \\
[4] &= (2n-3) P_1 + P_3 \\
[3,2] &= -(\tfrac{1}{2}n^2 - \tfrac{13}{2}n + 8) P_1 - 4P_3 + P_{2,1} \\
[2,2,2] &= -(\tfrac{1}{2}n^2 + \tfrac{9}{2}n + 5) P_1 + \tfrac{10}{3} P_3 - \tfrac{3}{2} P_{2,1} + \tfrac{1}{6} P_{1,1,1}
\end{aligned}$$

Remarks:

1. In general, the polynomials ϕ guaranteed by theorem 5.3 are not unique. For example, in the group algebra of S_3, $[3] = P_2 - 3P_0 = (P_{11} - 3P_0)/3$. However for fixed μ and n sufficiently large ($n > |\lambda|$) or "generic", then ϕ is unique.

2. More precisely, the proof shows that if we allow only terms P_μ such that $|\mu| \leq \text{rank}(\lambda)$ and $|\mu^+| \leq |\lambda|$, then the expansion of $[\lambda]$ in P_μ's is unique.

3. Richard Stong provided some helpful details in this argument.

6 Conclusion

For any finite group G, and any complete system of irreducible matrix representations, one obtains a maximal commutative subalgebra $\mathcal{M}(G) \subset \mathbf{Q}G$ which is the inverse image of diagonal matrices under Fourier transform. Conversely, an easy argument shows that $\mathcal{M}(G)$ determines the representation up to conjugation by a monomial matrix, that is, up to renormalization of the basis elements.

Thus the algebra $\mathcal{M}(S_n)$ generated by Murphy's elements is in a sense "equivalent" to Young's seminormal form (as well as Young's orthogonal form, which is obtained from the seminormal form by renormalization).

Since the R_i's have a simple intrinsic description, one can ask whether there are similar descriptions of $\mathcal{M}$ for other representations of S_n, and for other groups.

A conversation with Barry Mazur suggested the following illustration for D_n, the dihedral group with $2n$ elements, with n odd. This group has two irreducible representations of degree 1 and $(n-1)/2$ irreducible representations of degree 2. Hence every maximal commutative subalgebra of $\mathbf{Q}D_n$ has dimension $n+1$. One can construct examples of such subalgebras explicitly as follows: take the n elements of Z_n, viewed as a subgroup of D_n, together with any central element of $\mathbf{Q}D_n - \mathbf{Q}Z_n$, for example $\sum_{g \in D_n} g$. These elements generate a commutative subalgebra of the required dimension.

A similar construction works when n is even.

References

1. L. Chihara, On the zeroes of Askey-Wilson polynomials, with applications to coding theory, S.I.A.M. Jour. Math. Anal. **18** (1987) 183-207.

2. P. Diaconis, *Group Representations in Probability and Statistics*, Inst. Math. Statistics, Hayward CA, 1988.

3. P. Diaconis, Applications of non-commutative Fourier analysis in probablity problems, in Lecture Notes in Math **1362**, Springer-Verlag, Berlin, 1989, 51-100.

4. P. Diaconis and R.L. Graham, The Radon transform on Z_2^k, Pacific Journal of Math. **118** (1985), 323-345.

5. P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, Zeit. für Wahrscheinlichkeitstheorie und verwandte Gebiete **57** (1981), 453-476.

6. C.F. Dunkl, Structures isomorphic to compact homogeneous spaces, unpublished, 1972.

7. L. Flatto, A. Odlyzko, D. Wales, Random shuffles and group representations, Annals of Probability **13** (1985), 154-178.

8. G. Frobenius, Über die charaktere der symmetrischen Gruppe, Sitzungsberichte Königl. Preuss. Akad. Wissenshaften (Berlin, 1900), 516-534.

9. A. Greenhalgh, Random walks on groups with subgroup invariance properties, Thesis, Stanford University, 1989.

10. I.I. Hirschman, Integral equations on certain compact homogeneous spaces, S.I.A.M. Jour. Math. Anal. **3** (1974), 314-343.

11. R. E. Ingram, Some characters of the symmetric group, Proc. Amer. Math. Soc. **1** (1950), 358-369.

12. G. James, *The representation theory of the symmetric groups*, Lecture Notes in Math. **682**, Springer Verlag, Berlin, 1978.

13. G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, 1981

14. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford U. Press, 1979.

15. P. A. MacMahon, Memoir on the theory of the partitions of numbers — Part IV, Philosophical Transactions R.S. **A-209** (1909), 153-175.

16. P. Matthews, Covering problems for random walks on spheres and finite groups, Ph.D. thesis, Stanford University, 1985.

17. G. E. Murphy, A New Construction of Young's Seminormal Representation of the Symmetric Group, Jour. Algebra **69** (1981), 287-297.

18. O. Rothaus and J.G. Thompson, A combinatorial problem in the symmetric group, Pacific J. Math. **18** (1966), 175-178.

19. J. P. Serre, Lectures on constructive Galois theory, notes, Harvard University, 1989.

20. R. P. Stanley, Theory and application of plane partitions, Part II, Studies in Applied Math. **1** (1971), 259-279.