Mathematics 136, **Final Winter 2008**

Write your name and sign the Honor code in the blue books provided.

This is a closed material exam, but you may use six pages (2 sides each) of notes. You have 180 minutes to solve all **four** questions, each worth points as marked (maximum of 100). Complete reasoning is required for full credit. You may cite lecture notes and homework sets, as needed, stating precisely the result you use, why and how it applies.

1.(5x5) Provide the complete, accurate and rigorous definitions for the following **five** concepts.

(a) Pointwise convergence, almost sure convergence, and convergence in probability of a sequence of random variables $\{X_n\}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ to another R.V. X_∞ on this space.

ANS: Pointwise convergence happens when $X_n(\omega) \to X_\infty(\omega)$for each $\omega \in \Omega$. For a.s. convergence and convergence in probability detail Definitions 1.3.1 and 1.3.5.

(b) A version of a stochastic process X_t, a modification of a stochastic process X_t and the canonical filtration of X_t.

ANS: Definitions 3.1.8, 3.1.9 and 4.1.3.

(c) A right-continuous filtration $\mathcal{F}_t$, a square-integrable martingale $(M_t, \mathcal{F}_t)$ with continuous sample path, and the increasing process A_t associated with it.

ANS: Definitions 4.2.1 for a martingale $(M_t, \mathcal{F}_t)$ and 4.2.10 for a right-continuous filtration. The S.P. $\{M_t\}$ is square-integrable if $\mathbf{E}M_t^2$ is finite for all t and has continuous sample path if $t \mapsto M_t(\omega)$ is a continuous function for all $\omega \in \Omega$. For the increasing process A_t associated with it detail Theorem 4.4.7 and Definition 4.4.8.

(d) The Brownian motion W_t and the Geometric Brownian motion Y_t.

ANS: The Geometric Brownian motion is $Y_t = e^{W_t}$ and for the Brownian motion detail Definition 5.1.1.

(e) A Markov chain $\{X_n\}$, a homogeneous Markov chain and its strong Markov property.

ANS: Detail Definition 6.1.1, Definition 6.1.2 and (6.1.1).

2. (4+4+4+9+4+4) *Parts a)–f) of this problem are independent and can be solved independently of each other!*

Consider independent Brownian motions X_t and Y_t, the filtration $\mathcal{F}_t = \sigma(X_s, Y_s, s \le t)$ and the process $R_t = X_t^2 + Y_t^2$.

(a) Which, if any, of the stochastic processes X_t and R_t is a Gaussian process and which if any is a stationary process?

ANS: From Definition 5.1.1 we know that X_t is a Gaussian process. Note that $R_1 \ge 0$ and $\mathrm{Var}(R_1) > 0$. Such R.V. can not be Gaussian (check Proposition 3.2.10), so the process $\{R_t\}$ is not Gaussian. Further, $\mathbf{E}R_t = 2\mathbf{E}X_t^2 = 2t$ is non-constant, so both processes are non-stationary.

(b) Compute the probability that $X_2 > X_1$.

ANS: This is the probability that the increment $X_2 - X_1$ is positive. Since the latter is Gaussian of zero mean and positive variance, the probability in question is $1/2$.

(c) Provide a formula for the probability that $X_t < 4$ for all $0 \le t \le 3$.

ANS: You are asked to write $\mathbf{P}(\tau_4 > 3)$ for τ_α of (5.2.2). As shown there, this is merely $1 - 2\int_{4/\sqrt{3}}^{\infty} \phi(x)dx$ for the standard normal density $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$.

(d1) Find a non-random b such that $M_t = R_t - bt$ is a martingale with respect to $\mathcal{F}_t$ and prove that R_t is a sub-martingale.

ANS: Recall Exercise 4.2.5 that both $X_t^2 - t$ and $Y_t^2 - t$ are martingales. Because $\{X_t\}$ and $\{Y_t\}$ are independent, they both are also MGs for the filtration $\mathcal{F}_t$. The sum of two martingales with respect to same filtration $\mathcal{F}_t$ is a martingale for $\mathcal{F}_t$ (just check the definition of MG). Thus, we deduce that $(R_t - 2t, \mathcal{F}_t)$ is a martingale. This implies that $R_t - 2t$ is also a MG for its canonical filtration (for example, see Exercise 4.1.5). Adding a non-decreasing, non-random function to a MG leads to a sub-MG (just check the definitions of MG and sub-MG, see also the remark at end of Section 4.2). This of course applies to $R_t = (R_t - 2t) + 2t$.

(d2) Fixing $r > 0$ show that $\theta_r = \inf\{t \ge 0 : R_t \ge r^2\}$ is a stopping time with respect to $\mathcal{F}_t$.

ANS: The process R_t is adapted to $\mathcal{F}_t$ and has continuous sample-path (since both X_t and Y_t do). The set $B = [r^2, \infty)$ is closed, so this is a special case of Proposition 4.3.13(b).

(d3) Explain why $\theta_r \le \tau_r$ for $\tau_r = \inf\{t \ge 0 : |X_t| \ge r\}$ implies that $\mathbf{E}\theta_r$ is finite, then find the value of $\mathbf{E}\theta_r$.

ANS: Clearly, if $|X_t| \ge r$ then $R_t \ge r^2$, hence $\theta_r(\omega) \le \tau_r(\omega)$ for all $\omega \in \Omega$. By the monotonicity of the expectation, this implies that $\mathbf{E}\theta_r \le \mathbf{E}\tau_r$. Since you have shown in Exercise 5.2.5 that $\mathbf{E}\tau_r = r^2$ if finite (take $\alpha = \beta = r$ there), this shows that θ_r is also integrable (and in particular finite a.s.). We apply Doob's optional stopping (Theorem 4.3.17) to the martingale $(M_t, \mathcal{F}_t)$ of (d1) that has continuous sample path and the stopping time θ_r of (d2). In doing so, note that $|M_t| \le R_t + 2t$, so $|M_{t\wedge\theta_r}| \le r^2 + 2\theta_r := Y_r$. You have just proved that Y_r is integrable, so $\{M_{t\wedge\theta_r}\}$ is U.I. (recall Example 1.4.24). Thus,

$$0 = \mathbf{E}M_0 = \mathbf{E}M_{\theta_r} = \mathbf{E}R_{\theta_r} - 2\mathbf{E}\theta_r = r^2 - 2\mathbf{E}\theta_r\,,$$

resulting with $\mathbf{E}\theta_r = r^2/2$.

(e) For a Poisson process N_t of rate $\lambda > 0$, $s \ge t$ non-negative and n a non-negative integer compute the value of $v(n) = \mathbf{E}[N_s | N_t = n]$ and of $\mathbf{E}[N_s | N_t]$.

ANS: Recall that $N_s = (N_s - N_t) + N_t$ with $N_s - N_t$ a Poisson R.V. of parameter $\lambda(s-t)$ which is independent of N_t. Thus, $v(n) = n + \mathbf{E}[N_s - N_t] = n + \lambda(s-t)$. Consequently, $\mathbf{E}[N_s|N_t] = v(N_t) = N_t + \lambda(s-t)$.

(f) Repeat part (e) in case of $s < t$.

ANS: Recall Exercise 6.2.13(a) that conditional on $N_t = n$ the law of N_s is Binomial(n,p) for $p = s/t$. The latter has mean $np = ns/t$. Thus, in this case $v(n) = sn/t$ and $\mathbf{E}[N_s|N_t] = (s/t)N_t$ (which for $s = t$ coincides with part (e)).

3. (4+5+5) This problem is taken from your homework. You should solve it again. Merely citing the relevant homework solution will not earn points.

(a) Show that if $\mathbf{E}[X^2] = 0$ then $X = 0$ almost surely

Exercise 1.2.40, HW2: For $n \in \mathbb{N}$, let $A_n = \{|X| > 1/n\}$. Note that $\{X \neq 0\} = \cup_n A_n$. Hence by countable subadditivity it suffices to show that $\mathbf{P}(A_n) = 0$ for all n. This follows immediately by applying Markov's inequality (Theorem 1.2.38) to the function $f(x) = x^2$:

$$\mathbf{P}(A_n) \leq n^2 \mathbf{E}[X^2] = 0.$$

(b) Let $Z = (X, Y)$ be a uniformly chosen point on $(0,1)^2$. That is, X and Y are independent random variables, each having the $U(0,1)$ measure. Set $T = I_A(Z) + 5I_B(Z)$ where $A = \{0 < x < 1/4, 3/4 < y < 1\}$ and $B = \{3/4 < x < 1, 0 < y < 1/2\}$. Find an explicit formula for the conditional expectation $W = \mathbf{E}(T|X)$ and use it to determine the conditional expectation $U = \mathbf{E}(TX|X)$

Exercise 2.3.16(a), HW4: Note $A = A_1 \times A_2$ for $A_1 = \{x \in (0, 1/4)\}, A_2 = \{y \in (3/4, 1)\}$ hence $I_A(x,y) = I_{A_1}(x) I_{A_2}(y)$. Similarly $I_B(x,y) = I_{B_1}(x) I_{B_2}(y)$ for $B_1 = \{x \in (3/4, 1)\}, B_2 = \{y \in (0, 1/2)\}$. Consequently, $T = I_{A_1}(X) I_{A_2}(Y) + 5 I_{B_1}(X) I_{B_2}(Y)$. Thus, by the linearity of the C.E. and "taking out what is known" (Proposition 2.3.15) we have that

$$W = \mathbf{E}(T|X) = I_{A_1}(X)\mathbf{E}(I_{A_2}(Y)|X) + 5 I_{B_1}(X)\mathbf{E}(I_{B_2}(Y)|X).$$

Further, since X and Y are independent, $I_{A_2}(Y)$ and $I_{B_2}(Y)$ are independent of X. Thus, we have that

$$\mathbf{E}(I_{A_2}(Y)|X) = \mathbf{E} I_{A_2}(Y) = \mathbf{P}(Y \in A_2) = \frac{1}{4},$$

with the right-most identity due to Y being uniformly chosen on $(0,1)$ with A_2 an interval of length $1/4$. Similarly, $\mathbf{E}(I_{B_2}(Y)|X) = 1/2$, so we have that

$$W = \frac{1}{4} I_{A_1}(X) + \frac{5}{2} I_{B_1}(X).$$

Since X is bounded, we know that $U = \mathbf{E}(TX|X) = XW$ by Proposition 2.3.15.

(c) Fix $H \in (0,1)$. A Gaussian stochastic process $\{X_t,\ t \geq 0\}$ is called a fractional Brownian motion (or in short, fBM), of Hurst parameter H if $\mathbf{E}(X_t) = 0$ and

$$\mathbf{E}(X_t X_s) = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}],\ s, t \geq 0.$$

Show that an fBM of Hurst parameter H has a continuous modification that is also locally Hölder continuous with exponent γ for any $0 < \gamma < H$.

Exercise 5.1.11(a), HW6: Fix $0 < \gamma < H$. We have for all t, s and any positive integer n that

$$\mathbf{E}|X_t - X_s|^{2n} = C_n(\mathbf{E}|X_t - X_s|^2)^n = C_n(\mathbf{E}X_t^2 + \mathbf{E}X_s^2 - 2\mathbf{E}X_t X_s)^n = C_n|t-s|^{2Hn},$$

where C_n are some non-random finite constants (c.f. the explicit formula for moments of a normal random variable, immediately after the proof of Proposition 5.1.3). So from Kolmogorov's continuity theorem (with $\alpha = 2n$ and $\beta = 2Hn - 1$) we see that X_t possesses a continuous modification with any Hölder exponent in $(0, (2Hn-1)/2n)$. With $(2Hn-1)/2n = H-1/(2n)$ we get the desired result by taking n large enough so that $H - \frac{1}{2n} > \gamma$.

4.(3x10) State which of the following statements is true and which is false. You get 1 point for each correct answer (-1 point for each wrong answer) + 2 extra points for the correct reasoning (that is, citing the lecture notes, deriving from a known result or providing a counter example).

(a) If $\{X_t : t \geq 0\}$ is a Gaussian process and $s(t)\colon [0,\infty) \to [0,\infty)$ is a non-random function, then $\{X_{s(t)} : t \geq 0\}$ is also a Gaussian process.

True. Since X_t is a Gaussian process, Definition 3.2.17 tells you that the random vector $(X_{s(t_1)}, X_{s(t_2)}, \ldots, X_{s(t_n)})$ has a Gaussian distribution for all $t_1, t_2, \ldots, t_n \geq 0$. This is turn assures you that $\{Y_t : t \geq 0\}$ is a Gaussian process.

(b) The canonical filtration $\mathcal{G}_t$ of a stochastic process X_t is the same as the canonical filtration $\mathcal{H}_t$ of the stochastic process $Y_t = e^{X_t}$.

True. The mapping $x \mapsto e^x$ is invertible, so Y_t is adapted to $\mathcal{G}_t$ and $X_t = \log Y_t$ is adapted to $\mathcal{H}_t$. The former implies that $\mathcal{H}_t \subseteq \mathcal{G}_t$ while the latter implies that $\mathcal{G}_t \subseteq \mathcal{H}_t$ (see Definition 4.1.3). Consequently, $\mathcal{H}_t = \mathcal{G}_t$ as claimed.

(c) Any martingale is both a sub-martingale and a super-martingale.

True. See Remark 4.1.19.

(d) If $\tau = \inf\{n \geq 0 : S_n = 10\}$ for a symmetric simple random walk S_n on the integers (that is, $\mathbf{P}(S_1 = 1) = \mathbf{P}(S_1 = -1) = 1/2$), then $\mathbf{E}[S_{n\wedge\tau}] \to 10$ as $n \to \infty$.

False. Since S_n is a martingale (Example 4.1.8) and τ is a stopping time (Exercise 4.3.4), it follows that $S_{n\wedge\tau}$ is also a martingale (Theorem 4.3.6). The expectation of a martingale is constant (Remark 4.1.21), and $S_{0\wedge\tau} = S_0 = 0$, implying that $\mathbf{E}[S_{n\wedge\tau}] = 0$ for all n, so can't approach 10 in the limit $n \to \infty$.

(e) If a Gaussian stochastic process X_t of continuous sample path is a martingale with respect to its canonical filtration then $\lim_{t\to\infty} X_t$ exists w.p.1. and is integrable.

False. The Brownian motion W_t is a Gaussian martingale of continuous sample path and $\limsup_{t\to\infty} W_t$ is infinite w.p.1. because we have seen (between (5.2.1) and (5.2.2)) that for any $\alpha > 0$ the path W_t reaches level α at finite time (τ_α there). Since $-W_t$ has the same law as W_t, also $\liminf_{t\to\infty} W_t = -\infty$ w.p.1. so W_t does not converge as $t \to \infty$.

(f) If W_t is a Brownian motion and $\alpha > 0$ then $\frac{1}{\alpha} W_{\alpha^2 t}$ is also a Brownian motion.

True. See Exercise 5.1.4(d).

(g) $\mathbf{E}[1/(1 + |W_t - W_s|)] = \mathbf{E}[1/(1 + |W_{t-s}|)]$ for the Brownian motion W_t and any $t > s$.

True. We know that $W_t - W_s$ has same law as W_{t-s}, so for every $f(\cdot)$ we have $\mathbf{E}f(W_t - W_s) = \mathbf{E}f(W_{t-s})$. We simply took $f(x) = 1/(1 + |x|)$.

(h) With a positive probability there exist two random rational numbers $0 \leq r(\omega) < q(\omega) \leq 1$ such that the sample path of the Brownian motion $t \mapsto W_t(\omega)$ is monotone increasing in the open interval $r < t < q$.

False. Comparing Example 5.3.11 and Proposition 5.3.12 we have for any fixed $0 \leq a < b \leq 1$ that $\mathbf{P}(W_t(\omega)$ is monotone increasing in $t \in [a, b]) = 0$. Taking the union over the countably many rational numbers in $[0, 1]$ we see that the claim must be false.

(i) If X_t is a continuous time Markov process then so is $Y_t = X_t^2$.

False. Consider a process on $[0, 3]$ with two sample path, each occurring with probability one half, with $X_t(\alpha) = 1 + \mathbf{1}_{t\geq 2}$ and $X_t(\beta) = -\mathbf{1}_{t\geq 1}$. Since $X_t(\alpha) \geq 1$ and $X_t(\beta) \leq -1$, we see that there is no uncertainty in the value of X_{t+u} once X_t is given, so this is a Markov process (see Definition 6.1.10). In contrast, $Y_t(\alpha) = 1 + \mathbf{3}_{t\geq 2}$ and $Y_t(\beta) = \mathbf{1}_{t\geq 1}$ are of different values when $t < 1$ or $t \geq 2$ but are the same when $1 \leq t < 2$. Thus, while there is uncertainty in the value of $Y_{5/2}$ given $Y_{3/2}$ this uncertainty disappears once we also condition on the value of $Y_{1/2}$, showing that Y_t is not a Markov process.

(j) If N_t and M_t are independent Poisson processes of parameters λ and μ, respectively, then $N_t + M_t$ is a Poisson process of parameter $\lambda + \mu$.

True. See Proposition 6.2.17.