Statistics 310B, Winter 2009 **Final**

Write your name and sign the Honor code in the blue books provided.

You have 4 hours to solve the 3 questions in the exam and collect a maximum of 107 points (10 points for each part of questions 1 and 3, 9 points for each part of question 2).

Show all work and reasoning. Cite without proving any result that is proved in lecture notes or homework solutions, saying precisely where it is taken from, why and how it applies. If you can't solve one part of a question, skip it and feel free to use it when solving later parts.

All random variables are real valued, and on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

1. Fixing $s > 0$, the independent variables Z_n are such that $\mathbf{P}(Z_n = -1) = \mathbf{P}(Z_n = 1) = n^{-s}/2$ and $\mathbf{P}(Z_n = 0) = 1 - n^{-s}$. Starting at $Y_0 = 0$, for $n \geq 1$ let

$$Y_n = n^s Y_{n-1}|Z_n| + Z_n I_{\{Y_{n-1}=0\}}\,.$$

a). Is $\{Y_n\}$ a Markov chain and if so is a homogeneous Markov chain?

b). Show that $\{Y_n\}$ a martingale and that for any $x > 0$,

$$\mathbf{P}(\max_{k=1}^{n} Y_k \geq x) \leq \frac{1}{2x}[1 + \sum_{k=1}^{n-1}(k+1)^{-s}(1 - k^{-s})]\,.$$

c). Determine the values of $s > 0$, if any, for which $Y_n \to 0$ in probability, those for which also $Y_n \to 0$ almost surely and those for which $Y_n \to 0$ in L^1. You are to prove here both when convergence holds and when it does not.

d). Determine the values of $s > 0$, if any, for which $\{Y_n\}$ is L^1 bounded and those for which it is also uniformly integrable.

2. Consider a fair game in which at each successive bet you independently either gain an amount equal to your wager (with a probability half) or lose it. To determine your wagers, start with a finite sequence $x_1, x_2, \ldots, x_k$ of non-random positive numbers. Wager an amount that equals the sum of the first and last terms in the sequence. If you won your bet, delete those two numbers from your sequence. If you lost, append their sum as an extra term $x_{k+1} = x_1 + x_k$ at the right-hand end of the sequence. You play iteratively according to this rule till your sequence is empty (and if your sequence ever consists of one term only, you wager that amount, so upon wining you delete this term, while upon losing you append it to the sequence to obtain two terms). Let S_n denote the sum of terms and N_n the number of terms in your sequence after n turns of the game.

a). Show that with probability one you terminate with a profit $v = \sum_{i=1}^{k} x_i$ at some finite random time τ.

b). Show that τ is further an *integrable stopping time* with respect to the canonical filtration $\mathcal{F}_n$ associated with the game.

c). Denoting the maximal aggregate loss prior to termination by L, show that $\mathbf{E}L$ is infinite.

3. Consider the homogeneous Markov chain $Z_n = Z_{n-1} + \xi_n g(Z_{n-1})$ on the set of all integers, where $\{\xi_n\}$ are i.i.d. with $\mathbf{P}(\xi_1 = 1) = 1 - \mathbf{P}(\xi_1 = -1) = b$ and $g(z) = 1$ when $z \geq 0$, while $g(z) = -1$ for $z < 0$.

a). Show that $\{Z_n\}$ is irreducible if and only if $0 < b < 1$ and determine the period of each state of the chain in case $0 < b < 1$.

b). Show that $\{Z_n\}$ has a unique invariant measure (up to multiplication by a constant), when $0 < b \leq 1/2$ and determine for which of these values of b the chain further has an invariant probability measure.

With $\{U_n\}$ i.i.d. uniform on $[-5, 5]$ and independent of $\{\xi_n\}$ consider now the homogeneous Markov chain $\{X_n\}$ on the set of all *real numbers* where $X_n = X_{n-1} + \xi_n g(X_{n-1})$ when $|X_{n-1}| > 5$ and $X_n = X_{n-1} + U_n$ otherwise.

c). Show that $\{X_n\}$ is strongly H-irreducible for any $0 \leq b < 1$.

d). Show that if $0 < b < 1/2$ then $\{X_n\}$ has a unique invariant probability measure $\pi(\cdot)$ and that $\mathbf{P}_x(X_n \in B) \to \pi(B)$ as $n \to \infty$ for any $x \in I\!R$ and every Borel set B.