Final Exam: Solution

SOLUTION. Problem 1.

(a) Let $\mathcal{G}_n = \sigma(Y_k, k \le n)$. Clearly, Z_{n+1} is independent of $\mathcal{G}_n$. Hence, for any $n \ge 0$ and Borel set B the R.C.P.D. $\widehat{\mathbf{P}}_{Y_{n+1}|\mathcal{G}_n}(B, \omega)$ of Y_{n+1} given $\mathcal{G}_n$ is of the form $p_n(Y_n(\omega), B)$ for the Borel function

$$p_n(y, B) = \mathbf{P}[(n+1)^s y|Z_{n+1}| \in B]\mathbf{1}_{y \ne 0} + \mathbf{P}(Z_{n+1} \in B)\mathbf{1}_{y=0}\,.$$

Consequently, $\{Y_n\}$ is a Markov chain on $(\mathbb{R}, \mathcal{B})$, see Definition 6.1.2. It is not a homogeneous chain since for example $p_n(0, \{0\}) = \mathbf{P}(Z_{n+1} = 0) = 1 - (n+1)^{-s}$ depends on n.

(b) Clearly, Y_n is adapted with respect to the canonical filtration $\mathcal{G}_n$. Further, since $|Z_n| \le 1$ we see that $|Y_n| \le n^s|Y_{n-1}| + 1$. Thus, $\mathbf{E}|Y_n| \le n^s \mathbf{E}|Y_{n-1}| + 1$ for all $n \ge 1$ and with $\mathbf{E}|Y_0| = 0$ it follows by induction that $\{Y_n\}$ is integrable. Proceeding to verify the martingale condition, by the independence of Y_{n-1} and Z_n we find upon taking out what is known, that

$$\mathbf{E}(Y_n|Y_{n-1}) = n^s Y_{n-1}\mathbf{E}|Z_n| + I_{\{Y_{n-1}\ne 0\}}\mathbf{E}Z_n = Y_{n-1}$$

since $\mathbf{E}Z_n = 0$ and $\mathbf{E}|Z_n| = n^{-s}$.

Next, applying Doob's inequality for this martingale yields the bound $x^{-1}\mathbf{E}[(Y_n)_+]$. To compute the latter expectation, note that for any $n \ge 1$

$$(Y_n)_+ = n^s (Y_{n-1})_+|Z_n| + (Z_n)_+ I_{\{Y_{n-1}=0\}}\,.$$

Consequently, with $\mathbf{E}(Z_n)_+ = n^{-s}/2$ we have by the tower property and taking out what is known, that

$$\begin{aligned}\mathbf{E}[(Y_n)_+] &= \mathbf{E}[\mathbf{E}\{(Y_n)_+|Y_{n-1}\}] = n^s\mathbf{E}[(Y_{n-1})_+]\mathbf{E}|Z_n| + \mathbf{E}(Z_n)_+\mathbf{P}(Y_{n-1}=0) \\ &= \mathbf{E}[(Y_{n-1})_+] + \frac{1}{2}n^{-s}\mathbf{P}(Y_{n-1}=0)\,.\end{aligned}$$

Next observe that $A_n = \{\omega : Y_n(\omega) = 0\} = \{\omega : Z_n(\omega) = 0\}$ for any $n \ge 1$ and $A_0 = \Omega$ by our choice of $Y_0 = 0$. Consequently, $\mathbf{E}(Y_1)_+ = 1/2$ and with $\mathbf{P}(Y_k = 0) = (1 - k^{-s})$ for all $k \ge 1$, we get the stated bound by a simple recursive computation (or via induction).

(c) First note that $\mathbf{P}(Y_n \ne 0) = \mathbf{P}(A_n^c) \to 0$ as $n \to \infty$ for any $s > 0$. Therefore $Y_n \to 0$ in probability for any $s > 0$. Turning to convergence a.s., note that if $Y_n \ne 0$ then $|Y_n|$ is a finite product of terms, each one of which is at least one. Thus, $Y_n \to 0$ for $n \to \infty$ if and only if $\{Y_n \ne 0$ finitely often $\}$. Recall our observation that the events A_n are independent, hence by the pair of Borel-Cantelli lemmas, $\mathbf{P}(Y_n \ne 0$ finitely often $) = 1$ if and only if $\sum_n \mathbf{P}(A_n^c) < \infty$. That is, convergence a.s. of $\{Y_n\}$ happens if and only if $s > 1$.

Finally, recall that we have seen in part (b) that $\mathbf{E}(Y_n)_+$ is strictly increasing, hence does not converge to zero, regardless of the value of $s > 0$. It follows then that Y_n does not converges in L^1, regardless of $s > 0$.

(d) Recall Exercise 5.3.3 that L^1 boundedness of the martingale $\{Y_n\}$ is equivalent to the boundedness in n of $\mathbf{E}(Y_n)_+$. In part (b) we have seen that this is merely the convergence of the series $\sum_{k\geq 1}(k+1)^{-s}(1-k^{-s})$, so L^1-boundedness of $\{Y_n\}$ applies if and only if $s > 1$. Clearly, $\{Y_n\}$ is never U.I. since it never converges in L^1 (see Theorem 5.3.11, or alternatively, use Theorem 1.3.49).

Solution. Problem 2 = Exercise 5.4.10 of notes.

(a) Clearly, $S_0 = v$ and your profit at the end of n turns is merely $S_0 - S_n$, where a negative profit stands for your loss at that time. The game is fair and the sequence $\{V_n\}$ of wagers is $\mathcal{F}_n$-predictable and of bounded terms (since $V_k \leq 2^k \max(x_i, i \leq k)$ for all k). So, $(S_n, \mathcal{F}_n)$ is a martingale transform of $\{V_n\}$ with respect to the symmetric SRW associated with the game (assigning $+1$ to bets you lost and -1 to those you won). Thus, by Doob's convergence theorem, the non-negative martingale S_n converges a.s. when $n \to \infty$ to some non-negative $S_\infty \geq 0$ of finite mean. Of course, this implies that $|S_{n+1} - S_n| \stackrel{a.s.}{\to} 0$. However, each wager you make has to be at least for $\delta = \min(x_i, i \leq k) > 0$. Consequently, if $S_n > 0$ then $|S_{n+1} - S_n| \geq \delta$ and therefore if $S_n > 0$ for infinitely many values of n, then

$$\limsup_{n\to\infty} |S_{n+1} - S_n| \geq \delta \,.$$

In view of the preceding, we conclude that $\mathbf{P}(S_n > 0 \text{ finitely often }) = 1$, i.e. after an a.s. finite random time $\tau = \tau(\omega)$ we have that $S_\tau = S_\infty = 0$, resulting with profit v.

(b) With $\{N_n\}$ adapted to the canonical filtration $\{\mathcal{F}_n\}$ of the game, the corresponding first hitting time $\tau = \inf\{n \geq 0 : N_n < 1\}$, which is precisely when you terminate, is an $\mathcal{F}_n$-stopping time. Further, till time τ we can represent N_n as a random walk on the integers, starting at $N_0 = k$ and having transition probabilities $\mathbf{P}(N_{\ell+1} = x+1|N_\ell = x) = \mathbf{P}(N_{\ell+1} = x-2|N_\ell = x) = 1/2$ (corresponding to the probability that you lost or won your current bet, respectively). Clearly then $\mathbf{P}(\tau > n) \leq \mathbf{P}(N_n \geq 1)$ is summable, for $\mathbf{P}(N_n \geq 1) \leq c(k)b^n$ for some $c(k) < \infty$ some $b < 1$ and all n. Indeed, by Markov's inequality, for any $\eta > 0$,

$$\mathbf{P}(N_n \geq 1) \leq e^{-\eta}\mathbf{E}[e^{\eta N_n}] = e^{\eta(k-1)}b^n \,,$$

where $b = (e^\eta + e^{-2\eta})/2 < 1$ for all $\eta > 0$ small enough. Thus, as claimed $\mathbf{E}\tau$ is finite.

(c) By definition $L = \max\{S_n - S_0 : n = 0, 1, \ldots, \tau\}$, so $L + v \geq S_{n\wedge\tau} \geq 0$ for all $n \geq 0$. We see that if $\mathbf{E}L$ is finite then $\{S_{n\wedge\tau}\}$ is U.I. and by Doob's optional stopping theorem, $v = \mathbf{E}S_0 = \mathbf{E}S_\tau = 0$. Having arrived at a contradiction, we conclude that necessarily $\mathbf{E}L = \infty$.

Solution. Problem 3.

(a) The transition probability of the chain $\{Z_n\}$ of countable state space is $p(x, x+1) = 1-p(x, x-1) = b$ for $x \geq 0$ and $p(x, x+1) = 1-p(x, x-1) =$

$1-b$ for $x<0$. In case $0<b<1$ we have that $p(x,x\pm 1)>0$ for all x hence $\rho_{xy}>0$ for any integers x and y. By definition, the homogeneous chain $\{Z_n\}$ is then irreducible. In contrast, if $b=0$ we have $\rho_{01}=0$ and if $b=1$ we have $\rho_{10}=0$, so no irreducibility in either case. As for the period of the chain, note that $\mathbf{P}_0(X_n=0)=0$ whenever n is odd. Hence, $d_0\geq 2$ and actually $d_0=2$ since $\mathbf{P}_0(X_2=0)>0$. With the chain $\{Z_n\}$ irreducible, recall Lemma 6.2.50 that it has the same period at each state.

(b) In view of part (a) and Proposition 6.2.28 uniqueness (up to a constant multiple) of the invariant measure follows upon showing that zero is a recurrent state of $\{Z_n\}$ when $0<b\leq 1/2$. Starting at $X_0=0$ we have $X_1\in\{-1,1\}$ so by the symmetry of $p(x,y)$ for $x\neq 0$ under $(x,y)\mapsto(-x,-y)$ it suffices to show that $\mathbf{P}_{-1}(\tau_0<\infty)=1$. Setting $S_n=X_n+1$, this is merely the conclusion of part (d) of Exercise 5.4.7 (whose solution given as Exam practice). You can next easily check that $\mu(x)=\mu(-x-1)=v^x$, $x\geq 0$, satisfies the detailed balance relation $\mu(x)p(x,y)=\mu(y)p(y,x)$ for $v=b/(1-b)$ and all integers x,y, hence is a reversible measure for $\{Z_n\}$ (see Definition 6.2.31). Recall that a reversible measure is invariant. It is further easy to verify that $\sum_{x\in\mathbb{Z}}\mu(x)$ is finite when $0<b<1/2$ (i.e. $v<1$) and infinite when $b=1/2$. Consequently, $\{Z_n\}$ has a unique invariant probability measure for $0<b<1/2$ but not for $b=1/2$.

Alternatively, in case $b=1/2$ we merely have here the symmetric SRW on $\mathbb{Z}$ which is a null recurrent chain (per Example 6.2.41 of the notes), and has a unique invariant measure $\mu(x)=1$ which is not a finite measure. In case $b<1/2$ we apply the method of Exercise 6.2.44 for the non-negative Lyapunov function $h(x)=|x|$ which is zero at state $z=0$. Indeed, it is not hard to check that $(ph)(x)=p(x,x+1)|x+1|+p(x,x-1)|x-1|=|x|-\delta$ for $\delta=1-2b>0$ and all $x\neq 0$. Consequently, $z=0$ is a (strongly) positive recurrent state of $\{Z_n\}$, per part (c) of Exercise 6.2.44. By part (a) then this chain is positive recurrent, so has a unique invariant probability measure.

(c) We set $C=[-4,4]$ and let $q(\cdot)$ denote the uniform measure on $[-1,1]$ (many other choices of C and $q(\cdot)$ are of course possible). Then, it is easy to see that $p(x,B)=\lambda(B\cap[x-5,x+5])/10\geq\delta q(B)$ for $\delta=1/5$ and all $x\in C$ (where $\lambda(\cdot)$ denotes Lebesgue's measure on $\mathbb{R}$). Thus for strong H-irreducibility it suffices to verify that C is an accessible set of the chain $\{X_n\}$. To this end, observe that for $x\notin A=[-5,5]$ the chain $\{X_n\}$ has the same transition probabilities as $\{Z_n\}$. Consequently, from part (a) we deduce that $\mathbf{P}_x(\tau_A<\infty)>0$ for all $x\in\mathbb{R}$ (note that this applies also for $b=0$ in which case $\tau_A=\lceil(|x|-5)_+\rceil$ is still finite). With A accessible and $\mathbf{P}_x(X_1\in C)\geq 4/10>0$ for all $x\in A$, we conclude by the strong Markov property of the chain (at stopping time τ_A), that C is also an accessible set for it.

(d) The construction of part (c) has $q(C)=1$ which implies in particular aperiodicity of the chain $\{X_n\}$ (see Definition 6.3.26). In view of Theorems 6.3.24 and 6.3.27 it thus suffices to show that $\mathbf{P}_x(T_{\boldsymbol{\alpha}}<\infty)=1$ for all $x\in\mathbb{R}$ and that $\mathbf{E}_q T_{\boldsymbol{\alpha}}<\infty$.

Turning to the first task, recall part (c) that $\inf_{x\in A}\mathbf{P}_x(\overline{X}_2=\boldsymbol{\alpha}) = 1-\gamma>0$, hence by part (c) of Exercise 6.1.17 it suffices to show that $\mathbf{P}_x(T_A<\infty)=1$ for all $x\in\mathbb{R}$. As a matter of fact, considering separately the case of $X_1\in A$ and $X_1\notin A$, by the Markov property it suffices to show this for all $x\notin A$. But, outside A the chain $\{X_n\}$ evolves exactly as $\{Z_n\}$ of part (a) does. In particular, by irreducibility of the latter chain, $\mathbf{P}_x(T_A<\infty)=\rho_{y,\pm 5}=1$ where $x=\pm(y-\Delta)$ with $y\in\{6,7,\ldots\}$ and $0\le\Delta<1$.

Having established this, we proceed to our final (and in this case hardest) task of proving positive H-recurrence, namely the finiteness of $\mathbf{E}_q T_{\boldsymbol{\alpha}}$. To this end, consider the stopping times $r_k=\inf\{n\ge r_{k-1}+2 : \overline{X}_n\in A\}$, starting at $r_0=0$, which are a.s. finite in view of the preceding proof that $\mathbf{P}_x(T_A<\infty)=1$ for all $x\in\mathbb{R}$. Setting $\Gamma_k=\{\overline{X}_{r_k+2}\notin\boldsymbol{\alpha}\}$, clearly,

$$T_{\boldsymbol{\alpha}}\le 2+\sum_{k=1}^{\infty}(r_k-r_{k-1})\prod_{\ell=0}^{k-1} I_{\Gamma_\ell}\,.$$

With $0<\gamma<1$, by monotonicity of the expectation, the finiteness of $\mathbf{E}_q T_{\boldsymbol{\alpha}}$ thus follows upon showing that

$$\mathbf{E}_q[(r_k-r_{k-1})\prod_{\ell=0}^{k-1} I_{\Gamma_\ell}]\le c\gamma^k \qquad (*)$$

for some c finite and all $k\ge 1$. Next, setting $\mathcal{G}_n=\sigma(\overline{X}_k,k\le n)$, note that $|\overline{X}_{\widehat{\theta}}|<11$ at the $\mathcal{G}_n$-stopping time $\widehat{\theta}=r_{k-1}+2$ for any $k\ge 1$. Hence, by the strong Markov property of the chain

$$\mathbf{E}_q[(r_k-r_{k-1})|\mathcal{G}_{\widehat{\theta}}]=2+\mathbf{E}_{X_{\widehat{\theta}}}[\tau_A]\le 2+\sup_{|x|<11}\mathbf{E}_x[\tau_A]=c\,.$$

Further, for any $\ell\ge 0$, by the strong Markov property of the chain at the stopping time r_ℓ, in which the split chain is in A,

$$\mathbf{E}_q[I_{\Gamma_\ell}|\mathcal{G}_{r_\ell}]\le\gamma\,.$$

Since $\Gamma_\ell\in\mathcal{G}_{\widehat{\theta}}$ for all $\ell<k$, combining the last two bounds, we arrive at (*) upon iteratively applying the tower property k times and taking out what is known at each step.

It remains only to prove that $\sup_{|x|<11}\mathbf{E}_x[\tau_A]$ is finite. To this end it obviously suffices to consider only $|x|>5$ and then by symmetry to further restrict to $5<x<11$. As already seen before, in this range of x values we can further replace $\tau_A=T_A$ by $\widehat{T}_5=\inf\{n\ge 1: Z_n=5\}$ and the real valued $5<x<11$ by the integer valued $y\in\{6,7,\ldots,11\}$. Next observe that for such y values, forcing the first $y-5$ steps of $\{Z_n\}$ to be to the right yields the inequality,

$$\mathbf{E}_5[\widehat{T}_5]\ge b^{y-5}\mathbf{E}_y[\widehat{T}_5]\ge b^6\sup_{6\le y\le 11}\mathbf{E}_y[\widehat{T}_5]\,,$$

hence the finiteness of

$$\sup_{6\le y\le 11}\mathbf{E}_y[\widehat{T}_5]=\sup_{|x|<11}\mathbf{E}_x[\tau_A]$$

is a consequence of the positive recurrence of $\{Z_n\}$, which we have established already in part (b).