Sparse Components of Images and Optimal Atomic Decompositions

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Abstract:

Recently, Field, Lewicki, Olshausen, and Sejnowski have reported efforts to identify the "Sparse Components" of image data. Their empirical findings indicate that such components have elongated shapes and assume a wide range of positions, orientations, and scales.

To date, Sparse Components Analysis (SCA) has only been conducted on databases of small (e.g. 16-by-16) image patches and there seems limited prospect of dramatically increased resolving power. In this article, we apply mathematical analysis to a specfic formalization of SCA using synthetic image models, hoping to gain insight into what might emerge from a higher-resolution SCA based on n by n image patches for large n but constant field of view.

In our formalization, we study a class of objects $\mathcal{F}$ in a functional space; they are to be represented by linear combinations of atoms from an overcomplete dictionary, and sparsity is measured by the ℓ^p norm of the coefficients in the linear combination. We focus on the class $\mathcal{F} = \textsc{Star}^\alpha$ of black-and-white images with the black region consisting of a starshaped set with α-smooth boundary. We aim to find an optimal dictionary, one achieving the optimal sparsity in an atomic decomposition uniformly over members of the class $\textsc{Star}^\alpha$.

We show that there is a well-defined optimal sparsity of representation of members of $\textsc{Star}^\alpha$; there are decompositions with finite ℓ^p norm for $p > 2/(\alpha+1)$ but not for $p < 2/(\alpha+1)$. We show that the optimal degree of sparsity is nearly attained using atomic decompositions based on the wedgelet dictionary.

Wedgelets provide a system of representation by elements in a dyadically-organized collections at all scales locations orientations and positions. The atoms of our atomic decomposition contain both coarse-scale dyadic 'blobs', which are simply wedgelets from our dictionary, and fine-scale 'needles', which are differences of pairs of wedgelets.

The fine-scale atoms used in the adaptive atomic decomposition are highly anisotropic and occupy a range of positions, scales, and locations. This agrees qualitatively with the visual appearance of empirically-determined sparse components of natural images. The set has certain definite scaling properties; for example, the number of atoms of length l scales as $1/l$, and, when the object has α-smooth boundaries, the number of atoms with anisotropy $\approx A$ scales as $\approx A^{\alpha-1}$.